Geometric Constructions and Structures Associated with Twistor Spinors on Pseudo-Riemannian Conformal Manifolds

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Abstract

The present thesis studies local geometries admitting twistor spinors on pseudo- Riemannian manifolds of arbitrary signature using conformal tractor calculus. Many local geometric classification results are already known for the Riemannian and Lorentzian case. However, one is motivated to study the conformally covariant twistor equation also in higher signatures in full generality because of its new relations to higher-order conformal Killing forms, the possibly more interesting shapes of the zero set and relations to other geometric structures such as projective geometry or generic 2-distributions as has been recently discovered.

To this end, we refine and extend the necessary machinery of first prolongation of conformal structures and conformal tractor calculus which allows a conformally-invariant description of twistor spinors as parallel objects. In this context, our first main theorem is a classification result for conformal geometries whose conformal holonomy group admits a totally degenerate invariant subspace of arbitrary dimension: They are characterized by the existence of Ricci-isotropic pseudo-Walker metrics in the conformal class. This closes a gap in the classification results for non-irreducibly acting conformal holonomy.

Based on this we are able to prove a partial classification result for conformal structures admitting twistor spinors. Moreover, we study the zero set of a twistor spinor using the theory of curved orbit decompositions for parabolic geometries. Generalizing results from the Lorentzian case, we can completely describe the local geometric structure of the zero set, construct a natural projective structure on it and show that locally every twistor spinor with zero is equivalent to a parallel spinor off the zero set. An application of these results in low-dimensional split-signatures leads to a complete geometric description of manifolds admitting non-generic twistor spinors in signatures $(3,2)$ and $(3,3)$ in terms of parallel spinors which complements the well-known analysis of the generic case.

Moreover, we apply tractor calculus for the construction of a conformal superalgebra naturally associated to a conformal spin structure. This approach leads to various results linking algebraic properties of the superalgebra to special geometric structures on the underlying manifold. It also exhibits new construction principles for twistor spinors and conformal Killing forms. Finally, we discuss a $Spin^c$-version of the twistor equation and introduce and elaborate on the notion of conformal $Spin^c$-geometry. Among other aspects, this gives rise to a new characterization of Fefferman spaces in terms of distinguished $Spin^c$-twistor spinors.

Zusammenfassung

Die vorliegende Arbeit untersucht lokale Geometrien, die Twistorspinoren zulassen auf pseudo-Riemannschen Mannigfaltigkeiten beliebiger Signatur. Für den Riemannschen- und den Lorentzfall sind schon viele lokale geometrische Klassifikationsresultate bekannt. Man wird jedoch dazu motiviert, die konform-kovariante Twistorgleichung auch in höheren Signaturen in voller Allgemeinheit zu studieren, da sich hier neue, interessante Beziehungen zu konformen Killingformen von höherem Grad ergeben, die Nullstellenmenge eine interessantere Struktur aufweist und es Beziehungen zu anderen geometrischen Strukturen, wie projektiver Geometrie oder generischen 2-Distributionen gibt, wie kürzlich herausgefunden wurde.
Hierzu entwickeln wir die benötigten Methoden, nämlich die erste Prolongation konformer Strukturen und das konforme Traktorkalkül, welche eine konform-invariante Beschreibung von Twistorspinoren als parallele Objekte ermöglichen, weiter. In diesem Zusammenhang ist unser erstes zentrales Resultat ein Klassifikationssatz für konforme Strukturen, deren Holonomiegruppen einen total ausgearteten Unterraum beliebiger Dimension invariant lassen. Diese lassen sich durch Ricci-isotrope pseudo-Walker-Metriken in der konformen Klasse charakterisieren. Dies schliesst eine Lücke in der Klassifikation nicht irreduzibel wirkender konformer Holonomiegruppen.
Hierauf aufbauend können wir einen partiellen Klassifikationssatz für konforme Strukturen mit Twistorspinoren beweisen. Weiterhin studieren wir die Nullstellenmenge eines Twistorspinors unter Nutzung der Theorie der Orbitzerlegungen für parabolische Geometrien. Wir verallgemeinern aus dem Lorentzfall bekannte Resultate und können die lokale geometrische Struktur der Nullstellenmenge vollständig beschreiben. Weiterhin konstruieren wir eine natürliche projektive Struktur auf der Nullstellenmenge und zeigen, dass lokal jeder Twistorspinor mit Nullstelle konform äquivalent zu einem parallelem Spinor ist. Eine Anwendung dieser Resultate auf niedrig-dimensionale Split-Signaturen führt zu einer vollständigen geometrischen Beschreibung von Mannigfaltigkeiten mit nicht-generischen Twistorspinoren in den Signaturen $(3,2)$ und $(3,3)$ durch parallele Spinoren, was die schon bekannte Analyse des generischen Falls komplementiert.
Darüberhinaus wenden wir das Traktorkalkül an, um einer konformen Spin- Mannigfaltigkeit auf naürliche Weise eine konforme Superalgebra zuzuordnen. Dieser Zugang führt zu verschiedenen Resultaten, die algebraische Eigenschaften dieser Superalgebra mit speziellen Geometrien auf der zugrundeliegenden Mannigfaltigkeit in Verbindung bringen. Weiterhin erhält man so neue Konstruktionsprinzipien für Twistorspinoren und konforme Killingformen. Zuletzt diskutieren wir eine $Spin^c$–Version der Twistorgleichung und führen den Begriff der konformen $Spin^c$-Geometrie ein. Unter anderem liefern spezielle $Spin^c$-Twistorspinoren eine neue Charakterisierung von Fefferman-Räumen.

Contents

Introduction

The Problem

Conformal (Killing) vector fields are classical objects in differential geometry and also naturally appear in physics. Their properties are often directly linked to interesting geometric structures on the underlying pseudo-Riemannian manifold (cf. [KR95, KR97a, KR97b]). As a generalization, so called conformal Killing forms were introduced and studied from a physics point of view in [Kas68]. Moreover, [Sem01] discusses global properties of conformal Killing forms on Riemannian manifolds and [Lei05] investigates conformal Killing forms satisfying further normalisation conditions which naturally arise when studying conformal transformations of Einstein manifolds.
Besides, there is a spinorial analogue of the conformal Killing equation for vector fields which naturally appeared in physics as well as pure mathematics and leads to the notion of conformal Killing spinors or twistor spinors[1], and the aim of this thesis is the study of pseudo-Riemannian geometries admitting twistor spinors. Let us make this notion more precise and then discuss some aspects which motivate us to study such field equations:
We consider a space- and time-oriented, connected pseudo-Riemannian spin manifold of signature (p,q). One can canonically associate to this setting (cf. [LM89, Bau81]) the real resp. complex spinor bundle S^g with its Clifford multiplication, denoted by $cl : TM \times S^g \to S^g$, and the Levi-Civita connection lifts to a covariant derivative ∇^{S^g} on this bundle. Besides the (geometric) Dirac operator D^g, there is another, complementary, conformally covariant differential operator acting on spinor fields, obtained by performing the spinor covariant derivative ∇^{S^g} followed by orthogonal projection onto the kernel of Clifford multiplication,

$$P^g : \Gamma(S^g) \overset{\nabla^{S^g}}{\to} \Gamma(T^*M \otimes S^g) \overset{g}{\cong} \Gamma(TM \otimes S^g) \overset{proj_{\ker\ cl}}{\to} \Gamma(\ker\ cl),$$

called the Penrose-or twistor operator. These operators naturally lead to the following types of spinor field equations: A real or complex spinor field $\varphi \in \Gamma(S^g)$ is called

parallel spinor	$\Leftrightarrow$	$\nabla^{S^g}\varphi = 0$,
harmonic spinor	$\Leftrightarrow$	$D^g\varphi = 0$,
Killing spinor	$\Leftrightarrow$	$\nabla^{S^g}_X\varphi = \lambda X \cdot \varphi,\ \lambda \in \mathbb{C}$,
twistor spinor	$\Leftrightarrow$	$P^g\varphi = 0$.

The local formula for P^g reveals that twistor spinors are equivalently characterized as solutions of the conformally covariant twistor equation

$$\nabla^{S^g}_X\varphi + \frac{1}{n}X \cdot D^g\varphi = 0 \text{ for all } X \in \mathfrak{X}(M).$$

Obviously, every parallel spinor is a twistor spinor, and a harmonic spinor is parallel if and only if it is a twistor spinor at the same time.

[1] Throughout this thesis we shall use the name twistor spinor rather than conformal Killing spinor.

The interplay between the existence of nontrivial solutions to the above natural spinor field equations and underlying special geometric structures has a long history in pure mathematics as well as in mathematical physics.
The existence of parallel spinor fields on a given manifold allows one to place field theories on it which preserve some supersymmetry, see [Far00]. For the Riemannian case, one can complete the classification of spin manifolds admitting parallel spinors in terms of the well-established theory of Riemannian holonomy groups. More precisely, [MS00] shows that on a given $n-$dimensional Riemannian manifold, spin structures with parallel spinors are in one to one correspondence with lifts to $Spin(n)$ of the Riemannian holonomy group, with fixed points on the spin representation space. In higher signatures, a complete holonomy classification is hindered by the fact that there may be totally degenerate holonomy-invariant subspaces. Nevertheless, [Kat99] presents a complete classification of all non-locally symmetric, irreducible pseudo-Riemannian holonomy groups admitting parallel spinors. The other extremal case to irreducibly acting holonomy is the case of a maximal holonomy invariant totally lightlike subspace. This leads to parallel pure spinors on pseudo-Riemannian manifolds which are studied in [Kat99]. In split signatures, an explicit normal form of the metric is known. Moreover, the interplay between parallel spinors and Lorentzian holonomy groups has been studied in [Lei04].
Also Killing spinors are directly linked to underlying geometric structures. T. Friedrich observed in [Fri80] that Killing spinors on a compact Riemannian spin manifold are related to the first eigenvalue of the geometric Dirac operator. C. Bärs celebrated correspondence (cf. [Bär93]) interprets real Killing spinors in terms of parallel spinors on the metric cone, which opens up a conceptual way to the classification of Riemannian manifolds admitting Killing spinors by making use of Riemannian holonomy theory. The case of imaginary Killing spinors on Riemannian manifolds has been solved in [BFGK91]. The Killing spinor equation has also been intensively studied in Lorentzian geometry, see [Boh99, Boh03], where certain imaginary Killing spinors can be used to characterize Lorentzian Einstein Sasaki structures. Furthermore, there are many examples and partial classification results for geometries admitting Killing spinors in any signature (cf. [Boh99, AC08, Kat99]). In mathematical physics, various generalizations of the geometric Killing equation appear in supergravity theories, recently also in the construction of supersymmetric Yang-Mills theories on curved space, see [AFHS98, MHY13], for instance.

Parallel- and Killing spinors are special examples of twistor spinors. The twistor equation in full generality first appeared in a purely mathematical context in [AHS78] as integrability condition for the natural almost complex structure of the twistor space of a Riemannian 4-manifold. In physics, twistor spinors appeared in the context of general relativity and were first introduced by R. Penrose in [PR86, Pen67]. They were studied from a local viewpoint and gave rise to integrability conditions in order to integrate the equations of motion. Since then, the twistor equation on Riemannian manifolds has been systematically studied, for instance in [BFGK91, Hab90, Hab94, Fri89]. Among other important classification results, it is well-known that a Riemannian spin manifold admitting a twistor spinor without zeroes is conformally equivalent to an Einstein manifold which admits a parallel or a Killing spinor. The zero set in the Riemannian case has been widely studied (cf. [Hab94, KR94]). It consists of isolated points, and if a zero exists, the spinor is conformally equivalent to a parallel spinor off the zero set.

Bearing the role of the twistor equation in physics literature in mind, H. Baum and F. Leitner started a systematic investigation of the twistor equation on arbitrary Lorentzian spin manifolds in [Bau99, Lei01, BL04, Lei07, Bau06]: A powerful tool is the observation that in the Lorentzian case, the so called Dirac current $V_\varphi \in \mathfrak{X}(M)$, defined by

$$g(V_\varphi, X) = -\langle X \cdot \varphi, \varphi \rangle_{S^g}$$

is a conformal vector field naturally associated to any twistor spinor φ whose zeroes coincide with that of φ. The most general classification result for Lorentzian geometries admitting twistor spinors was obtained in [Lei07]:

Theorem 0.1 ([Lei07]) *Let $\varphi \in \Gamma(S^g_{\mathbb{C}})$ be a twistor spinor on a Lorentzian spin manifold of dimension $n \geq 3$. Then on an open, dense subset of M one of the following holds, at least locally:*

1. *φ is locally conformally equivalent to a parallel spinor with lightlike Dirac current on a Brinkmann space.*
2. *(M, g) is locally conformally equivalent to $(\mathbb{R}, -dt^2) \times (N_1, h_1) \times \cdots \times (N_r, h_r)$, where the (N_i, h_i) are Ricci-flat Kähler, hyper-Kähler, G_2-or Spin(7)-manifolds.*
3. *φ is not locally conformally equivalent to a parallel spinor and exactly one of the following cases occurs:*
   a) *n is odd and there is (locally) a metric $\widetilde{g}$ in the conformal class such that $(M, \widetilde{g})$ is a Lorentzian Einstein-Sasaki manifold.*
   b) *n is even and (M, g) is locally conformally equivalent to a Fefferman space.*
   c) *There exists locally a product metric $g_1 \times g_2 \in [g]$ on M, where g_1 is a Lorentzian Einstein-Sasaki metric on a space M_1 admitting a Killing spinor and g_2 is a Riemannian Einstein metric with Killing spinor on a space M_2 of positive scalar curvature.*

Conversely, it is well-known that all special Lorentzian geometries appearing in Theorem 0.1 admit global solutions of the twistor equation, and these geometries have been intensively studied. Twistor spinors with zeroes in Lorentzian signature are studied in [Lei01, Lei07]. Analysing the zeroes of certain conformal vector fields, one deduces that the zero set of a twistor spinor with zero on a Lorentzian manifold consists either of isolated images of null-geodesics and off the zero set one has a parallel spinor on a Brinkmann space, or the zero set consists of isolated points and off the zero set one has a local splitting $(\mathbb{R}, -dt^2) \times (N, h)$, where the last factor is Riemannian Ricci-flat Kähler, in the conformal class. [Lei07] presents an example of a Lorentzian 5-manifold admitting a twistor spinor with isolated zero, where, however, it remains unclear whether the metric is also smooth at the zero.

In contrast to the Riemannian and Lorentzian case, the investigation of the twistor equation in other signatures is widely open. Let us elaborate in more detail on some aspects which may raise the interest in understanding the solutions of the twistor equation in arbitrary signature.

To start with, a new relation between projective and conformal geometry has recently been exhibited in [HS11a]: One starts with an oriented, torsion free projective structure and constructs a split-signature conformal structure on an open, dense subset of its cotangent bundle. This conformal structure is shown to admit twistor spinors, and one might use them to characterize such pseudo-Riemannian extensions, which have also been treated and appeared in [CGGV09, DW07]. Moreover, [HS11b] gives a spinorial characterization of 5-dimensional manifolds admitting the famous 2-distribution in dimension 5 of generic type, as introduced in [Nur05], in terms of so-called generic twistor spinors in signature $(3,2)$. The same can be carried out for split signatures in dimension 6 admitting a generic 3-distribution as considered in [Bry09]. Furthermore, manifolds of higher signature admitting twistor spinors produce natural examples for manifolds carrying (normal) conformal Killing forms of higher degree: To every twistor spinor φ on a pseudo-Riemannian manifold $(M^{p,q},g)$ one can associate a p–form α_{φ}^{p} which is trivial iff the spinor is trivial. The twistor equation translates into the conformal Killing equation plus additional normalisation conditions, see [Lei05], for α_{φ}^{p}. It is known that distinguished conformal Killing forms of higher degree can be used to equivalently characterize exceptional geometric structures such as nearly-Kähler manifolds in dimension 6 or nearly parallel G_2–manifolds in dimension 7, see [Bär93, Lei05, Sem01]. Finally, the index p of the underlying pseudo-Riemannian manifolds is an upper bound for the dimension of the zero set of a twistor spinor. Thus, twistor spinors on Riemannian or Lorentzian manifolds can only exhibit isolated points or images of curves as zero sets, as has also been showsn in [BFGK91, Lei01]. One can therefore expect more interesting possible shapes of the zero set and relations to other geometric structures in higher signatures.

These aspects of pseudo-Riemannian twistor spinor theory raise our interest in the subsequent differential-geometric questions:

1. *Which pseudo-Riemannian geometries admit nontrivial solutions of the twistor equation ?*
2. *How are further properties of twistor spinors related to the underlying geometries ? In particular, what are the possible shapes of the zero set $Z_{\varphi} \subset M$?*
3. *How can one construct examples of manifolds admitting twistor spinors ?*

A review of the known classification results for twistor spinors in arbitrary signature reveals that the answers to the above questions are widely open: A well-understood case are twistor spinors on Einstein spaces. [BFGK91] shows that in case of nonzero scalar curvature the spinor decomposes into a sum of two Killing spinors whereas in case of a Ricci-flat metric the spinor $D^g\varphi$ is parallel.
As there is no complete classification of manifolds admitting twistor spinors, one often restricts oneself to small dimensions in order to find out which geometries play a role there. [Bry00] classifies metrics admitting parallel spinor fields in all signatures that occur in small dimensions. It is moreover known that a Riemannian 3-manifold admitting a twistor spinor is conformally flat, and a Riemannian 4-manifold with twistor spinor is selfdual (cf. [BFGK91]). In Lorentzian geometry, there is a classification of all local geometries admitting twistor spinors without zeroes and constant causal type of the associated conformal vector field V_{φ} for dimensions $n \leq 7$, which can be found in [Lei01] or [BL04]. In signature $(2,2)$, anti-selfdual 4-manifolds with parallel real spinor have been studied

in [Dun02]. Furthermore, [HS11a] presents a Fefferman construction which starts with a 2-dimensional projective structure and produces geometries carrying two linearly independent twistor spinors. [HS11b] investigates (real) *generic* twistor spinors in signature $(3,2)$ and $(3,3)$, being twistor spinors satisfying additionally that the constant (!) $\langle\varphi, D\varphi\rangle \neq 0$ (signature $(3,3)$ is also discussed in [Bry09]). They are shown to be in tight relationship to so called generic 2-distributions on 5-manifolds resp. generic 3-distributions on 6-manifolds, meaning that the second commutator of these distributions already is TM. Every generic twistor spinor gives rise to a generic distribution, and conversely, given a manifold with generic distribution, one can canonically construct a conformal structure admitting a twistor spinor, and these two constructions are inverse to each other.

Besides these local geometric classification results, twistor spinors are also of interest from a slightly different point of view in conformal geometry and physics: In flat Minkowski space, the study of supersymmetry theories leads to the study of extensions of the Poincaré algebra to a superalgebra (cf. [BW92]). In curved space this generalizes to the following constructions: One considers the Lie algebra of Killing vector field and adds, as an odd part, infinitesimal spinorial symmetries, which are described by Killing spinors with respect to a suitable connection. This object can then be given the structure of a (Lie) superalgebra as discussed in [Far99]. Simple superalgebras and their classifications have been studied in [Nah78]. They are important in supergravity theories in physics whereas we believe that their possible mathematical role in classifying geometric structures has by far not been fully examined. There is a conformal analogue of this construction which also naturally appears in physics, but which we will consider for purely geometric reasons: [Hab96] observes that the space $\mathfrak{g}_0 = \mathfrak{X}^c(M)$ of conformal vector fields on a given pseudo-Riemannian manifold together with the space $\mathfrak{g}_1 = \ker P^g$ of twistor spinors carries a natural structure of a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. This construction has then also been studied in [Raj06, MH13]. The references present concrete examples which show that $\mathfrak{g}$ need not to be a Lie superalgebra. However, it still remains unclear in which way geometric structures on the underlying manifold (M,g) are encoded in algebraic properties of its conformal superalgebra.

Finally, we want to motivate the study of a generalization of the twistor equation to $Spin^c$–geometry. As outlined before, distinguished spinor fields φ on pseudo-Riemannian manifolds naturally induce by squaring distinguished vector fields or differential forms α^p_φ. More precisely, one finds

φ		α^p_φ
parallel spinor	$\Rightarrow$	parallel form,
Killing spinor	$\Rightarrow$	special Killing form,
twistor spinor	$\Rightarrow$	normal conformal Killing form.

Special Killing forms and normal conformal Killing forms are subspaces of Killing forms resp. conformal Killing forms, distinguished by further differential normalization conditions as discussed in [Sem01, Lei05]. In this context, it is natural to ask whether there is also a spinorial analogue of generic conformal Killing forms. As we shall see, this is indeed the case if one introduces an analogue of the twistor operator on $Spin^c$-manifolds which requires the inclusion of a S^1-connection as additional underlying datum.
More precisely, one starts with a space- and time-oriented, connected pseudo-Riemannian

$Spin^c$-manifold (M,g) of signature (p,q) with canonical underlying S^1-principal bundle $\mathcal{P}_1$. Associated to this setting are the complex spinor bundle S^g with its Clifford multiplication $cl : TM \otimes S^g \to S^g$. If moreover a connection A on $\mathcal{P}_1$ is given, there is a canonically induced covariant derivative ∇^A on S^g. As in the $Spin$-setting, besides the Dirac operator D^A, defined by composing ∇^A with cl, there is another conformally covariant and complementary differential operator acting on spinor fields, obtained by performing the spinor covariant derivative ∇^A followed by orthogonal projection onto the kernel of Clifford multiplication,

$$P^A : \Gamma(S^g) \overset{\nabla^A}{\to} \Gamma(T^*M \otimes S^g) \overset{g}{\cong} \Gamma(TM \otimes S^g) \overset{proj_{\ker\, cl}}{\to} \Gamma(\ker\, cl),$$

which we shall call the $Spin^c$-twistor operator. Elements of its kernel are equivalently characterized as solutions of the conformally covariant $Spin^c$-twistor equation

$$\nabla^A_X \varphi + \frac{1}{n} X \cdot D^A \varphi = 0 \text{ for all } X \in \mathfrak{X}(M), \tag{0.1}$$

whose solutions we call $Spin^c$–twistor spinors or charged conformal Killing spinors (CCKS).

Charged conformal Killing spinors are natural candidates for the spinorial analogue of conformal, not necessarily normal conformal Killing forms. Besides, there are further interesting purely geometric reasons for the study of (0.1). First, it is a natural generalization of $Spin^c$-parallel-and Killing spinors which have been investigated in [Mor97]. Their study has lead to new spinorial characterizations of Sasakian and pseudo-Kähler structures in the Riemannian case. Generalizations of the $Spin^c$–Killing spinor equations have been studied in [GN13]. Moreover, we have the hope that CCKS might lead to equivalent characterizations of manifolds admitting certain conformal Killing forms. By this, we mean the following. Given a CCKS φ, one can as in the $Spin$–setting always form its associated Dirac current V_φ. In the $Spin$–case, distinguished by $dA = 0$, V_φ is always a *normal* conformal vector field, i.e. it inserts trivially into the Cotton York- and Weyl tensor. However, for Lorentzian 3-manifolds it has been shown in [HTZ13] that locally for *every* zero-free conformal, not necessarily normal conformal lightlike or timelike vector field V there is a CCKS φ wrt. a generally non-flat connection A such that $V = V_\varphi$. The same holds on Lorentzian 4-manifolds for lightlike conformal vector fields, see [CKM+14]. We want to investigate whether this principle carries over also to other signatures. This would lead to spinorial characterizations of manifolds admitting certain conformal symmetries.
In addition to that, the $Spin^c$-version of the twistor equation has also appeared and been discussed from a physics point of view in [CKM+14, HTZ13, KTZ12]: Recently, it has become an interesting topic in mathematical physics to place certain supersymmetric Minkowski-space theories on curved space which may lead to new insights in the computation of observables, see [Pes12, FS11, CKM+14, HTZ13, KTZ12]. Requiring that the deformed theory on curved space preserves some supersymmetry again leads to generalized Killing spinor equations. Interestingly, one finds for different theories and signatures, namely Euclidean and Lorentzian 3-and 4 manifolds the same type of spinorial equation, namely a $Spin^c$-analogue of the twistor spinor equation, see for instance [CKM+14, HTZ13, KTZ12]. As shown in these references, one can derive this twistor equation also by using the AdS/CFT-correspondence and studying the gravitino-variation near the conformal boundary.

Let us now elaborate on the methods one uses in order to classify pseudo-Riemannian geometries admitting twistor spinors. They turn out to be of interest in their own right. The above classification results for twistor spinors on Riemannian manifolds can be obtained by explicit calculations (cf. [BFGK91]). In general, due to the conformal covariance of the twistor equation, twistor spinors are objects of conformal geometry, i.e. it makes sense to define twistor spinors if one is only given a conformal manifold $(M, c = [g])$, where $\widetilde{g}$ is equivalent to g iff $\widetilde{g} = f \cdot g$ for some positive function f. In an arbitrary pseudo-Riemannian context one thus has to make use of modern methods of conformal geometry as presented in [Feh05, CS09, BJ10]. One could describe objects of conformal geometry by defining them with respect to some metric and then show that the definition does not depend on the choice of this metric. This, however, often leads to long calculations and the geometric meaning remains in the dark.
A slightly different approach to objects of conformal geometry goes along the lines of the access to differential geometry in [Sha97]: The key object is the flat model for conformal structures, being the pseudo-Möbius sphere $(S^p \times S^q)/\mathbb{Z}_2$ equipped with the obvious conformal structure of signature (p, q), which generalizes the sphere with the conformal class of the round standard metric to arbitrary signatures (cf. [Feh05, CS09]). Every curved conformal structure should then locally look like this flat model. This notion is made precise by the first prolongation of a conformal structure which uses Cartan connections and methods of parabolic geometry. One ends up with a Cartan geometry $(\mathcal{P}^1, \omega^{nc})$ of type $(B := O(p+1, q+1), Stab_{\mathbb{R}^+ e_-} B)$, where $e_- \in \mathbb{R}^{p+1,q+1}$ is some lightlike vector, naturally associated to a given conformal structure of signature (p, q). The Cartan connection ω^{nc} is called the normal conformal Cartan connection. Passing to associated bundles leads to the standard tractor bundle equipped with a linear connection which can be viewed as the conformal analogue of the Levi-Civita connection. One then defines the conformal holonomy $Hol(M, c)$ of the conformal structure as being the holonomy of this connection and has that $Hol(M, c) \subset O(p+1, q+1)$.
Conformal holonomy groups are also of interest in their own right as they are basic invariants of conformal structures. Properties of the conformal holonomy representations are directly linked to special geometries in the conformal class such as almost Einstein scales, local splittings into Einstein products or Fefferman metrics, see [Arm07, BJ10, Lei07].
An analogue first prolongation procedure can then also be carried out in the conformal spin setting, and associated bundles to spinor representations lead to spin tractor bundles. In this language, one can equivalently describe twistor spinors as parallel sections in the spin tractor bundle associated to a conformal spin manifold as presented in [BJ10] or [Lei01]. In this setting, geometries admitting twistor spinors are equivalently characterized as those conformal spaces (M, c) where the lift of the conformal holonomy group $Hol(M, c) \subset SO(p+1, q+1)$ to $Spin(p+1, q+1)$ stabilizes a nontrivial spinor, see [Lei01, BJ10]. This is completely analogous to the description of parallel spinors on pseudo-Riemannian manifolds.
However, this very elegant method does not yet allow for a complete classification of pseudo-Riemannian conformal structures admitting twistor spinors as there is no complete classification of possible conformal holonomy groups: This problem is completely solved only in the Riemannian case (cf. [Arm07, BJ10, Lei06]). In arbitrary signatures, one knows a conformal analogue of the local de-Rham/Wu-splitting theorem (cf. [Lei07]) and all holonomy groups acting transitively and irreducibly on the Möbius sphere were

classified in [Alt12]. One difficulty which complicates a more general classification is that in contrast to the metric setting, where the holonomy algebra turns out to be a Berger algebra, the Ambrose-Singer Theorem does not lead to a useful algebraic criterion for $\mathfrak{hol}(M,c)$ which would allow the classification of irreducibly acting conformal holonomy algebras. As for the classification of metric holonomy groups, the most involved case is the situation when the holonomy representation fixes a totally lightlike subspace $H \subset \mathbb{R}^{p+1,q+1}$. The associated local geometries are only known in cases dim $H \leq 2$ ([BJ10, LN12a]).

The previous review makes clear that except from the Riemannian and Lorentzian case the following questions related to twistor spinors and their classifications are of interest and widely open:

1. *Which pseudo-Riemannian conformal holonomy groups $Hol(M,c)$ can occur in the presence of a twistor spinor. Which twistor spinors are* `true` *twistor spinors, i.e. not equivalent to parallel spinors, and is there a characterisation in terms of conformal holonomy ?*

2. *What are the possible shapes of the zero set of a twistor spinors and is every twistor spinor locally equivalent to a parallel spinor off the zero set (as it holds in the Riemannian and Lorentzian case) ?*

3. *What local geometries can occur in low dimensions in any signature. In particular, what is the geometric meaning of twistor spinors in signatures (3,2) and (3,3) where the associated 2-resp. 3-distribution is non-generic ?*

4. *Is the construction of a conformal superalgebra associated to a conformal spin structure also possible in the conformal tractor calculus, and what is the geometric meaning of this algebra ?*

5. *How can one formulate the twistor equation in the framework of conformal Spin^c-geometry. What is the precise relation of twistor spinors to conformal Killing forms in this situation ?*

To put it differently, we would like to understand the differential-geometric constructions and clarify the implications in the following diagram:

$$\begin{tikzcd}
& \alpha^k_\varphi \text{ conf. Killing form} & \\
& (M^{p,q},[g]) \text{ with } \varphi \in \ker P^g \arrow[u] \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{shape of } Z_\varphi & Hol(M,[g]) =? \arrow[l] \arrow[r] & \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \text{ superalgebra}
\end{tikzcd}$$

This thesis exhibits and reveals various interesting new relations between these objects of conformal geometry.

Outline of the thesis and results

In this thesis we provide (partial) answers to the above raised questions in arbitrary signature. We first study relevant aspects of spinor algebra and then describe the characterization of conformal structures in terms of Cartan geometries. Afterwards, we focus on the relation between twistor spinors and conformal holonomy groups. Based on this, we study the zero set of a twistor spinor and apply the results obtained so far in low dimensional split signatures up to dimension 12. We discuss a construction of a conformal superalgebra associated to a Lorentzian conformal structure by making use of tractor calculus. Finally, we study a $Spin^c$–version of the twistor equation. Let us describe the outline and the main results of this thesis in more detail:

The first two chapters contain well-known material. We do not seek completeness or elegance of the exposition as none of the stated results is unknown. However, we supply enough details to make the text accessible to readers not familiar with one of the techniques and refer to further literature for more comprehensive introductions.

In **chapter 1** we study relevant aspects of spinor algebra in arbitrary signature. Following [Bau81, LM89, Har90] we introduce Clifford algebras and spin groups, and as central object their representation spaces $\Delta_{p,q}$, denoting the real or complex spinor module. We further study scalar products and distinguished orbits on $\Delta_{p,q}$ and make precise the relation between $\Delta^{\mathbb{R}}_{p,q}$ and $\Delta^{\mathbb{C}}_{p,q}$, the real and complex spinor module. This allows a more uniform treatment when dealing with twistor spinors later, as there are authors which define spinor fields by only using the complex version (cf. [Bau81, Lei01]), whereas others focus on real spinor fields (cf. [AC97, Raj06]). Furthermore, we study associated forms α^k_φ to a spinor $\varphi \in \Delta_{p,q}$ for arbitrary $k \in \mathbb{N}$ which generalize the associated Dirac current from the Lorentzian case (cf. [Lei01, BL04]) to arbitrary signatures. We investigate algebraic properties of these forms which are later brought into connection with underlying geometric structures.

In **chapter 2** we provide the elaboration of the fundamental principles and methods to work on the classification problem for twistor spinors in pseudo-Riemannian geometry. We start with reviewing basic objects from principal bundle theory and Cartan geometry with a focus on holonomy groups of (Cartan) connections. The relevant objects of semi-Riemannian geometry and semi-Riemannian spin geometry are recalled and we introduce the differential operators acting on spinors which play a role in this thesis. We further list basic properties and integrability conditions for twistor spinors. An elegant way to describe twistor spinors as objects of conformal geometry is the conformal tractor calculus as developed in [BJ10] or [Feh05], which associates a natural Cartan geometry to a given conformal structure as described in the remainder of this chapter. There are different approaches to the first prolongation of a conformal structure and tractor calculus. We try to be as explicit as possible and follow [BJ10], and not the more general approach in [CS09] which makes use of parabolic geometries. One is then in position to introduce associated tractor (form) bundles, conformal holonomy groups and their spinorial analogues. Here, special emphasis is put on the description of associated tractor bundles and their covariant derivatives wrt. some fixed metric in the conformal class.

The chapters 3 to 7 present new results which (partially) answer the questions raised in the above introduction.

The methods of conformal tractor calculus enable us in **chapter 3** to state a precise relationship between twistor spinors and conformal holonomy groups. First, we outline how twistor spinors can be equivalently described as parallel sections in the conformal spin tractor bundle $\mathcal{S}(M)$, which is known from [Lei01]. Next, it is discussed how every twistor spinor gives rise to a nontrivial parallel tractor form, which via some fixed metric corresponds to a normal conformal Killing form as investigated in [Lei05]. These results reveal that the existence of twistor spinors leads to special properties of the conformal holonomy representation of the associated conformal structure. An easy application characterizes conformal structures admitting twistor spinors whose conformal holonomy representation acts irreducibly on $\mathbb{R}^{p+1,q+1}$ and transitively on the pseudo-Möbius sphere in Theorem 3.13.

Next, we observe in section 3.3 that every parallel spin tractor ψ gives rise to a (possibly trivial) totally lightlike, conformally invariant and parallel distribution, called ker ψ, in the standard tractor bundle, which is defined as the pointwise kernel of Clifford multiplication with ψ. This distribution becomes one of the main objects throughout this thesis and appears in various situations. It makes clear that the study of twistor spinors naturally leads to the study of conformal structures whose conformal holonomy representation fixes a totally lightlike subspace. Up to now, results are only known if the dimension of the totally lightlike holonomy-invariant subspace is ≤ 2. We generalize this and prove in Proposition 3.20 and Theorem 3.22:

Theorem 0.2 *If on a conformal manifold (M,c) there exists a totally lightlike, k-dimensional parallel distribution in the standard tractor bundle, then there is an open, dense subset $\widetilde{M} \subset M$ and a totally lightlike distribution $L \subset T\widetilde{M}$ of constant rank $k-1$ which is integrable. Moreover, every point in $\widetilde{M}$ admits a neighbourhood U and a metric $g \in c_U$ such that*

$$\begin{aligned} Ric^g(TU) &\subset L, \\ L_{|U} &\text{ is parallel wrt. } \nabla^g. \end{aligned} \tag{0.2}$$

Conversely, if $U \subset M$ is an open set equipped with a metric $g \in c_U$ and a $k-1$-dimensional totally lightlike distribution $L \subset TU$ such that (0.2) holds, then L naturally induces a $k-$dimensional totally lightlike, parallel distribution in the standard tractor bundle over U.

This statement closes a gap in the literature as together with other results (cf. [Arm07, Lei06, Lei07]) it allows a complete description of conformal structures admitting non-irreducibly acting conformal holonomy groups. Application of this result to twistor spinors leads to a description of all twistor spinors being equivalent to parallel spinors in terms of conformal holonomy and the associated distribution ker ψ in Propositions 3.28 and 3.34. Built on this, we can then prove the following partial classification results for conformal structures admitting twistor spinors in arbitrary signature (Theorem 3.43):

Theorem 0.3 *Let $\psi \in Par(\mathcal{S}(M), \nabla^{nc})$ be a parallel spin tractor on a conformal spin manifold (M,c) of signature (p,q) and dimension $n = p+q \geq 3$. For $g \in c$ let $\varphi \in ker P^g$ denote the associated twistor spinor. Exactly one of the following cases occurs:*

1. *It is ker* $\psi \neq \{0\}$. *In this case,* φ *can locally be rescaled to a parallel spinor on an open, dense subset* $\widetilde{M} \subset M$, *and ker* $\varphi \subset TM$ *is an integrable distribution on* $\widetilde{M}$. *In case that the respective metric holonomy acts irreducible and the space is not locally symmetric, it is one of the list in Theorem 3.2. Otherwise, one has a parallel spinor on a Ricci-isotropic pseudo Walker manifold. The conformal holonomy representation* $Hol(M,c)$ *is never irreducible but fixes a nontrivial totally lightlike subspace.*

2. *It is ker* $\psi = \{0\}$. *The spinor* φ *cannot be locally rescaled to a parallel spinor. Depending on the conformal holonomy representation, exactly one of the following cases occurs:*

   a) $Hol(M,c)$ fixes a totally lightlike subspace. *In this case, there is locally around each point a metric in the conformal class such that* φ *is a twistor spinor which is not Killing on a Ricci-isotropic pseudo Walker manifold. If* $Hol(M,c)$ *fixes an isotropic line, then there is a Ricci-flat metric* $g \in c$ *on which* $D^g\varphi$ *is nontrivial and parallel.*

   b) $Hol(M,c)$ acts reducible and fixes only nondegenerate subspaces. *In this case, there is around each point of an open, dense subset* $\widetilde{M} \subset M$ *an open neighbourhood* U *and a metric* $g \in c_U$ *such that either*

      - (U,g) *is an Einstein space and* φ *decomposes into the sum of two Killing spinors.*

      - $(U,g) \overset{isom.}{\cong} \pm dt^2 \times (V,g')$, *where the last factor is an Einstein space admitting a Killing spinor.*

      - $(U,g) \overset{isom.}{\cong} (H,h) \times (V,g')$, *where the first factor is a two dimensional space and* (V,g') *is an Einstein space admitting a Killing spinor.*

      - $(U,g) \overset{isom.}{\cong} (M_1,g_1) \times (M_2,g_2)$, *where* (M_i,g_i) *are Einstein spaces of dimensions* ≥ 3. (M_1,g_1) *admits a real Killing spinor to the Killing number* $\lambda \neq 0$ *and* (M_2,g_2) *admits an imaginary Killing spinor to* $i \cdot \mu$, *where* $|\lambda| = |\mu|$.

   c) $Hol(M,c)$ acts irreducible. *If the action on the conformal Möbius sphere is transitive, then* $Hol(M,c)$ *is one of the groups listed in Theorem 3.13. If there exists a metric* $g \in c$ *satisfying both* $C^g = 0$ *and* $\nabla W^g \neq 0$, *i.e.* (M,g) *is a Cotton space and not conformally symmetric, then* $Hol(M,c)$ *is one of the groups in Theorem 3.2.*

This analysis narrows the open cases in the classification problem for conformal geometries admitting twistor spinors to two very special geometric situations which remain open. Examples and generation principles for each case are discussed. The previous main Theorem may also be summarized more systematically in the following table, where the required notation and terminology will be developed during the chapter:

$\ker\psi$	$Hol(M,c)$	local geometry $g \in c$, behaviour of φ
$\neq \{0\}$	fixes $\ker\psi$	φ parallel on Ricci-isotropic pseudo-Walker metric
$\{0\}$	fixes totally lightlike subspace H	φ non-parallel on Ricci-isotropic pseudo-Walker metric
	• dim H=1	$D^g\varphi$ parallel
	• dim $H > 1$	$\omega \cdot D^g\varphi$ recurrent
	fixes only non-degenerate subspaces	Splitting into Einstein spaces admitting Killing spinors
	acts irreducibly on $\mathbb{R}^{p+1,q+1}$	
	• acts transitively on $Q^{p,q}$ or there is a non-conformally symmetric C-space in the conformal class	Fefferman spin space or S^3−bundle over quat. contact manifold with non-parallel twistor spinors, or generic cases in signatures $(3,2)$,$(3,3)$
	• does not act transitively on $Q^{p,q}$ and there is no C-space in the conformal class	*No example known*

Chapter 4 is then devoted to the study of the zero set of a twistor spinor which up to now is only completely described in the Riemannian and Lorentzian case. We can completely determine the zero set of twistor spinors on the homogeneous model in Proposition 4.5. Using the curved orbit decomposition for arbitrary Cartan geometries, which has recently been studied in [CGH14] we then completely describe the local structure of the zero set of arbitrary twistor spinors in Theorem 4.3:

Theorem 0.4 *Let $\varphi \in \Gamma(S^g)$ be a twistor spinor with zero $x \in M$. Then the zero set Z_φ is an embedded, totally geodesic and totally lightlike submanifold of dimension dim ker $D^g\varphi(x)$, where the last quantity does not depend on the choice of $x \in Z_\varphi$. Moreover, for every $x \in Z_\varphi$ there are open neighbourhoods U of x in M and V of 0 in T_xM such that*

$$Z_\varphi \cap U = exp_x\left(ker\ D^g\varphi(x) \cap V\right).$$

We discuss how this formula generalizes the known results from the Riemannian and Lorentzian case. Furthermore, we describe a natural way of constructing a projective structure on the zero set in Proposition 4.13:

Proposition 0.5 *Let $\varphi \in \Gamma(S^g)$ be a nontrivial twistor spinor with $Z_\varphi \neq \varnothing$ on (M,c). Then for every $g \in c$ the Levi-Civita connection ∇^g descends to a torsion-free linear connection ∇ on Z_φ. If g and $\widetilde{g}$ are conformally equivalent, the induced connections ∇ and $\widetilde{\nabla}$ are projectively equivalent, i.e., there is a natural construction*

$$\varphi \text{ on } (M,c) \to (Z_\varphi, [\nabla])$$

from conformal structures admitting a twistor spinor with zero to torsion-free projective structures on the zero set.

This Proposition opens up a new relation between projective and conformal geometry which we discuss in more detail. We also outline how the naturally induced projective

structure arises on the level of Cartan geometries.
Furthermore, we prove in Theorem 4.17 that every twistor spinor with zero can off the zero set locally be rescaled to a parallel spinor. In the Lorentzian case one can directly link the local geometry of the zero set to geometric structures off the zero set. We show how this can be generalized to conformal structures of index 2 in Proposition 4.21:

Proposition 0.6 *Let $\varphi \in \Gamma(S^g)$ be a twistor spinor with zero on $(M^{2,n-2}, g)$. Then exactly one of the following cases occurs:*

1. *Z_φ consists locally of* totally lightlike planes*. In this case, the spinor is locally equivalent to a parallel spinor off the zero set and gives rise to a parallel totally lightlike 2-form.*
2. *Z_φ consists of* isolated images of lightlike geodesics*. In this case, the spinor is off the zero set locally conformally equivalent to a parallel spinor on a Brinkmann space.*
3. *Z_φ consists of* isolated points*. In this case there is for each point off the zero set an open neighbourhood and a local metric in the conformal class such that the resulting space is isometric to a product $(U_1, g_1) \times (U_2, g_2)$, where the first factor is Ricci-flat pseudo-Kähler and the second factor (which might be trivial) is Riemannian Ricci-flat. Both factors admit a parallel spinor.*

In **chapter 5** we apply the previous results to real twistor spinors in low dimensions. Our classification theorems for conformal holonomy exhibit how essential information about possible local geometries admitting twistor spinors in signature (p,q) is encoded in the orbit structure of $\Delta^{\mathbb{R}}_{p+1,q+1}$ under action $Spin(p+1, q+1)$. It is with algebraic results from [Bry00, Igu70, GE78] then possible to present a complete list of possible local geometries for twistor (half)spinors in signatures (m,m) or $(m-1,m)$ with $m \leq 6$ in Theorem 5.1. Furthermore, we can give the possible local shapes of the zero set in each signature. In particular, this chapter complements the analysis of the generic cases from [HS11a] in signatures $(3,2)$ and $(3,3)$. We elaborate on this in more detail in section 5.4 and find together with the results from [HS11b]:

Theorem 0.7 *Let $(M,[g])$ be a conformal spin manifold of signature $(3,2)$ admitting a real twistor spinor $\varphi \in \Gamma(S^g)$. Then the function $\langle \varphi, D^g\varphi \rangle_{S^g}$ is constant and the value of this constant does not depend on the chosen metric in $[g]$. We distinguish the following cases:*

1. *$\langle \varphi, D^g\varphi \rangle_{S^g} \neq 0$. In this case, $Hol(M,[g]) \subset G_{2,2}$, the 2-dimensional distribution ker $\varphi \subset TM$ is generic and the whole conformal structure can be recovered from it.*
2. *$\langle \varphi, D^g\varphi \rangle_{S^g} = 0$. In this case, $Hol(M,[g])$ fixes a 3-dimensional totally lightlike subspace, there is an open, dense subset $\widetilde{M} \subset M$ on which the distribution ker φ is of constant rank 2 and integrable. Moreover, φ is locally conformally equivalent to a parallel pure spinor wrt. a local metric from Theorem 3.36 which lies in the conformal class.*

A similar statement is proved for signature $(3,3)$ in Theorem 5.6.

Chapter 6 is devoted to the construction of conformal superalgebras in Lorentzian signature. We show in section 6.1 that the space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of parallel spin tractors and parallel tractor 2-forms carries a natural superalgebra structure, which we call the tractor conformal superalgebra. Via some fixed metric in the conformal class, one obtains superalgebras isomorphic to those constructed in [Hab96, Raj06] as we show in section 6.2. However, we see that the tractor approach has the advantage of giving conditions when the construction actually leads to a Lie superalgebra in geometric terms. We prove:

Theorem 0.8 *Suppose that the conformal holonomy representation of (M,c) satisfies the following: There exists for $x \in M$* ***no*** *(possibly trivial) $m-$dimensional Euclidean subspace $E \subset \mathcal{T}_xM \cong \mathbb{R}^{2,n}$ such that both*

1. *The action of $Hol_x(M,c)$ fixes E,*
2. *On $E^{\perp}$, $Hol_x(M,c)_{E^{\perp}} := \{A_{|E^{\perp}} \mid A \in Hol_x(M,c)\} \subset O(E^{\perp}) \cong O(2,n-m)$ is conjugate to a subgroup of $SU(1,\frac{n-m}{2}) \subset SO(2,n-m)$.*

The the tractor conformal superalgebra satisfies the odd-odd-odd Jacobi identity, and thus carries the structure of a Lie superalgebra.

The construction of a tractor conformal superalgebra is discussed for various cases, including flat Minkowski space, small dimensions of $\mathfrak{g}_1$ and irreducible conformal holonomy. Furthermore, we show how the space $\mathfrak{g}$ can be turned into a Lie superalgebra in case of a generic Fefferman- or Lorentzian Einstein Sasaki metric in the conformal class, which are the geometries that do not meet the requirements from Theorem 0.8, under inclusion of an R-symmetry in section 6.4. This is inspired by considerations from [MH13]. Our main result which relates the structure of $\mathfrak{g}$ to local geometries on (M,c) is the following:

Theorem 0.9 *Let $(M^{1,n-1},c)$ be a Lorentzian conformal spin structure admitting twistor spinors. Assume further that all twistor spinors on (M,c) are of the same type according to Theorem 0.1. Then there are the following relations between special Lorentzian geometries in the conformal class c and properties of the tractor conformal superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of (M,c):*

Twistor spinor type (Thm. 0.1)	*Special geometry in c*	*Structure of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$*
1.	*Brinkmann space*	*Lie superalgebra*
2.	*Splitting $(\mathbb{R}, -dt^2)\times$ Riem. Ricci-flat*	*Lie superalgebra*
3.a	*Lorentzian Einstein Sasaki (n odd)*	*No Lie superalgebra, becomes Lie superalgebra under inclusion of nontrivial R-symmetry*
3.b	*Fefferman space (n even)*	*No Lie superalgebra, becomes Lie superalgebra under inclusion of nontrivial R-symmetry*
3.c	*Splitting $M_1 \times M_2$ into Einstein spaces*	*No Lie superalgebra, odd part splits $\mathfrak{g}^i = \mathfrak{g}_0^i \otimes \mathfrak{g}_1^i$, but $\mathfrak{g}_0 \neq \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$*

Finally, we outline how the construction of a tractor conformal superalgebra can be generalized to arbitrary signatures in section 6.6, where as a byproduct we present interesting new formulas that construct new normal conformal killing forms and twistor spinors out of existing ones in Propositions 6.34 and 6.39, where the latter Proposition generalizes the well-known spinorial Lie derivative. For instance, for every normal conformal Killing k−form $\alpha_+ \in \Omega^k(M)$ for (M,g) and every $\varphi \in kerP^g$, the spinor

$$\alpha_+ \circ \varphi := \frac{2}{n}\alpha_+ \cdot D^g\varphi + \frac{(-1)^k}{n-k+1} d^*\alpha_+ \cdot \varphi + \frac{(-1)^{k+1}}{k+1} d\alpha_+ \cdot \varphi \in \Gamma(S^g)$$

turns out to be again a twistor spinor on (M,g). We illustrate in more detail how this procedure of constructing new conformal forms and spinors out of existing ones works for the case of 6-dimensional nearly Kähler manifolds in section 6.7.

The final **Chapter 7** introduces and investigates the basic properties of the $Spin^c$-twistor operator whose definition additionally involves a S^1−connection. It is straightforward to derive integrability conditions relating the conformal Weyl curvature tensor W^g to the curvature dA of the S^1-connection. We introduce the notion of conformal $Spin^c$-structures and discuss them in the framework of Cartan geometries. Interestingly, we find in Theorem 7.11 that $Spin^c$-twistor spinors correspond to spin tractors on the first prolongation which are parallel wrt. a nontrivially modified Cartan connection. This can later be interpreted as a spinorial analogue of the description of conformal, not necessarily normal conformal Killing forms via the machinery of BGG-sequences and modified connections as known from [Ham08]. We furthermore show that the Dirac current associated to a generic $Spin^c$-twistor spinor is a conformal, in general not normal conformal vector field.
It is then natural to ask for construction principles of Lorentzian manifolds admitting global solutions of the $Spin^c$-twistor spinor equation. We are motivated by the following: Every simply-connected pseudo-Riemannian Ricci-flat Kähler spin manifold admits (at least) 2 parallel spinors, see [BK99]. Given a Kähler manifold equipped with its canonical $Spin^c$-structure and the S^1-connection A canonically induced by the Levi-Civita connection, [Mor97] shows that there is (generically) one $Spin^c$-parallel spinor wrt. A and $dA = 0$ iff the manifold is Ricci-flat. It is known that Fefferman spin spaces over strictly pseudoconvex manifolds can be viewed as the Lorentzian and conformal analogue of Calabi-Yau manifolds and that they always admit 2 conformal Killing spinors. This construction is presented in detail in [Bau99] and from a conformal holonomy point of view in [BJ10, Lei07]. In view of this, it is natural to conjecture that there is a $Spin^c$-analogue. Indeed, we find in Theorem 7.23 that every Fefferman space (F^{2n+2}, h_θ) over a strictly pseudoconvex manifold (M^{2n+1}, H, J, θ) admits a canonical $Spin^c$-structure and a natural S^1-connection A on the auxiliary bundle induced by the Tanaka Webster connection on M such that there exists a $Spin^c$-twistor spinor on F. Under additional natural assumptions also the converse direction is true, leading to a new characterization of Fefferman spaces in terms of $Spin^c$-spinor equations in Theorem 7.25:

Theorem 0.10 *Let $(B^{1,2n+1}, h)$ be a Lorentzian $Spin^c$-manifold. Let $A \in \Omega^1(\mathcal{P}_1, i\mathbb{R})$ be a connection on the underlying S^1-bundle and let $\varphi \in \Gamma(S^g)$ be a nontrivial CCKS wrt. A such that*

1. *The Dirac current $V := V_\varphi$ of φ is a regular isotropic Killing vector field,*

2. $V \lrcorner W^h = 0$ *and* $V \lrcorner C^h = 0$*, i.e.* V *is a normal conformal vector field,*

3. $V \lrcorner dA = 0$,

4. $\nabla^A_V \varphi = ic\varphi$*, where* $c = const \in \mathbb{R}\backslash\{0\}$.

Then (B, h) *is a* S^1*-bundle over a strictly pseudoconvex manifold* (M^{2n+1}, H, J, θ) *and* (B, h) *is locally isometric to the Fefferman space* (F, h_θ) *of* (M, H, J, θ).

Further, we obtain a classification of local Lorentzian geometries admitting CCKS under the additional assumption that the associated conformal vector field is normal conformal in Theorem 7.27. Our study of the $Spin^c$–twistor equation on Lorentzian 5-manifolds leads to a equivalent spinorial characterization of geometries admitting Killing 2-forms of a certain causal type in Theorem 7.37. We obtain similar results in signatures $(0, 5), (2, 2)$ and $(3, 2)$.
Finally, we study the general relation between generic conformal Killing forms and normal conformal Killing forms as considered in [Lei05], by elaborating on some illuminating examples. They reveal that, under additional assumptions, the difference between normal conformal Killing forms and conformal Killing forms corresponds to passing from conformal spin geometry to conformal $Spin^c$-geometry on the spinorial level.

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1 Spinor Algebra

The aim of this chapter is the study of spinor modules $\Delta_{p,q}$ in any signature (p,q). To this end, we introduce Clifford algebras as well as Spin groups and then study the associated real spinor module $\Delta^{\mathbb{R}}_{p,q}$ and its complex counterpart $\Delta^{\mathbb{C}}_{p,q}$. The results obtained in this chapter allow a more uniform treatment of twistor spinors in the real and complex case in what follows. Main references are [Bau81, Har90, LM89].

1.1 Pseudo-Euclidean space and its orthogonal transformations

Let $\mathbb{R}^{p,q}$ denote the real vector space $\mathbb{R}^n$, where $n = p+q$, equipped with a scalar product[1] $\langle \cdot,\cdot \rangle_{p,q}$ of signature (p,q), satisfying that

$$\langle e_i, e_j \rangle_{p,q} = \epsilon_i \delta_{ij},$$

where $\epsilon_i \in \{\pm 1\}$ and $(e_1,...,e_n)$ is the standard basis of $\mathbb{R}^n$. In general, we should think of the standard pseudo-Euclidean scalar product, defined by $\epsilon_i = -1$ iff $i = 1,...,p$. However, it turns out be useful to work with this more general notion. Any basis $(a_1,...,a_n)$ of $\mathbb{R}^{p,q}$ satisfying $\langle a_i, a_j \rangle_{p,q} = \epsilon_i \delta_{ij}$ will be called a (pseudo-)orthonormal basis of $\mathbb{R}^{p,q}$. For any $x \in \mathbb{R}^{p,q}$, we let $x^\flat := \langle x, \cdot \rangle_{p,q} \in (\mathbb{R}^{p,q})^*$ denote the dual wrt. $\langle \cdot,\cdot \rangle_{p,q}$. In particular, we note that $e_i^\flat(e_j) = \epsilon_i \delta_{ij}$, yielding an isomorphism $\flat : \mathbb{R}^{p,q} \to (\mathbb{R}^{p,q})^*$ with inverse map denoted by $\sharp$. With respect to the standard basis and its dual, row vectors can be identified with elements of $(\mathbb{R}^{p,q})^*$, column vectors with elements of $\mathbb{R}^{p,q}$, and in this picture the isomorphisms $\sharp$ and $\flat$ are given by $z^\sharp = J_{p,q} z^t$ and $x^\flat = x^t J_{p,q}$, where $J_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$, and I_k is the identity matrix with k rows. $\langle \cdot,\cdot \rangle_{p,q}$ induces scalar products on the dual space, tensor products and, in particular, on the space of r-forms $\Lambda^r (\mathbb{R}^{p,q})^* =: \Lambda^r_{p,q}$ in a natural way and also denoted by $\langle \cdot,\cdot \rangle_{p,q}$. We will work with the scalar product given by [2]

$$\langle e^\flat_{k_1} \wedge ... \wedge e^\flat_{k_r}, e^\flat_{l_1} \wedge ... \wedge e^\flat_{l_r} \rangle_{p,q} := \det\left((\langle e_{k_i}, e_{l_j} \rangle_{p,q})_{ij} \right).$$

Remark 1.1 If $\mathbb{R}^{p,q}$ is enlarged to $\mathbb{R}^{p+1,q+1}$ it is convenient to do the following: We fix the pseudo-orthonormal standard basis $(e_0, e_1, ..., e_n, e_{n+1})$ of $\mathbb{R}^{p+1,q+1} \cong \mathbb{R}^{1,1} \oplus \mathbb{R}^{p,q}$, where $(e_1,...,e_n)$ is the standard basis of $\mathbb{R}^{p,q}$ and e_0 is a timelike (i.e. $\langle e_0, e_0 \rangle_{p+1,q+1} < 0$) and e_{n+1} is a spacelike vector. We introduce two lightlike[3] directions by setting $e_\pm := \frac{1}{\sqrt{2}}(e_{n+1} \pm e_0)$. It is convenient to work with the basis $(e_-, e_1, ..., e_n, e_+)$ of $\mathbb{R}^{p+1,q+1} \cong \mathbb{R}e_- \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}e_+$.

[1] In this thesis, a scalar product on a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ is a nondegenerate and symmetric resp. Hermitian bilinear resp. sesquilinear form.

[2] In this thesis, our convention for the wedge product of $\omega \in \Lambda^k_{p,q}$ and $\sigma \in \Lambda^l_{p,q}$ is $\omega \wedge \sigma(x_1,...,x_{k+l}) := \frac{1}{k! \cdot l!} \sum_{\pi \in S_{k+l}} \operatorname{sgn} \pi \cdot \omega(x_{\pi(1)},...,x_{\pi(k)}) \cdot \sigma(x_{\pi(k+1)},...,x_{\pi(k+l)})$.

[3] In this thesis, vectors in $\mathbb{R}^{p,q}$ with $\langle v, v \rangle_{p,q} = 0$ are called either lightlike or isotropic or null.

With respect to this basis, the Gram matrix of $\langle\cdot,\cdot\rangle_{p+1,q+1}$ takes the form $\begin{pmatrix}0 & 0 & 1\\ 0 & J_{p,q} & 0\\ 1 & 0 & 0\end{pmatrix}$. In particular, we observe that $\langle e_-, e_+\rangle_{p+1,q+1} = 1$ and $\langle e_\pm, e_k\rangle_{p+1,q+1} = 0$ for all $1 \le k \le n$.

Let $O(p,q)$ denote the Lie group $O(\mathbb{R}^{p,q})$, i.e. the subgroup of $GL(\mathbb{R}^{p,q})$[4] consisting of all elements that preserve $\langle\cdot,\cdot\rangle_{p,q}$. Following [Bau81] the group $O(p,q)$ has four connected components for $0 < p < n$: Choosing an orthonormal basis for $\mathbb{R}^{p,q}$, we can represent elements $A \in O(p,q)$ as $A = \begin{pmatrix}T_1 & X\\ Y & T_2\end{pmatrix}$, where $T_1 \in GL(p,\mathbb{R}), T_2 \in GL(q,\mathbb{R})$ and the four connected components can be shown to be

$$O^{\epsilon_1,\epsilon_2}(p,q) := \{A \in O(p,q) \mid \text{sgn det } T_1 = \epsilon_1, \text{sgn det } T_2 = \epsilon_2\} \text{ with } \epsilon_{1,2} \in \{\pm 1\}.$$

Furthermore, we set $SO(p,q) := O^{++}(p,q) \cup O^{--}(p,q)$ and $SO^+(p,q) := O^{++}(p,q)$ which is the connected identity component of $O(p,q)$.

With respect to the standard basis of $\mathbb{R}^n$, the matrices $E_{kl} := \epsilon_k D_{lk} - \epsilon_l D_{kl}$ with $k < l$ form a basis of the Lie algebra $\mathfrak{o}(p,q) \subset \mathfrak{gl}(n,\mathbb{R})$ of $O(p,q)$. Here, D_{kl} denotes the matrix in $M(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$ whose (k,l) entry is 1 and all other entries are 0. The Lie algebra relations read

$$[E_{ij}, E_{kl}]_{\mathfrak{o}(p,q)} = \begin{cases} 0 & i = k, j = l \text{ or } i,j,k,l \text{ pairwise distinct,}\\ \epsilon_i E_{jl} & i = k, j \neq l, \end{cases} \tag{1.1}$$

from which all other relations follow with $E_{ij} = -E_{ji}$. There is, moreover, a natural vector space isomorphism

$$\Theta : \Lambda^2_{p,q} \to \mathfrak{so}(p,q),\ \Theta(\alpha)(x) = (x \lrcorner \alpha)^\sharp, \tag{1.2}$$

whose inverse satisfies $\Theta^{-1}(A)(x,y) = (Ax)^\flat(y)$. Under this isomorphism, the basis vectors $e_k^\flat \wedge e_l^\flat$ of $\Lambda^2_{p,q}$ and E_{kl} of $\mathfrak{so}(p,q)$ for $k < l$ are identified, i.e. $\Theta\left(e_k^\flat \wedge e_l^\flat\right) = E_{kl}$.

In the sequel, certain representations of $SO^+(p,q)$ will become of important. More generally, we introduce the following notation for a Lie group G and $\rho : G \to GL(V)$ a representation of G over a real or complex vector space V carrying a scalar product. The dual representation ρ^* of G on the dual space V^* is denoted by

$$\rho^*(g) := [\rho(g^{-1})]^T \ \forall g \in G,$$

where A^T stands for the transpose of a linear map A. A representation ρ^{*r} of G on $\Lambda^r(V^*)$ is induced by $\rho^{*r}(g)(\sigma^1 \wedge \ldots \wedge \sigma^r) := \rho^*(g)(\sigma^1) \wedge \ldots \wedge \rho^*(g)(\sigma^r)\ \forall g \in G, \sigma^i \in V^*$. The stabilizer of $v \in V$ (under the G-action ρ) is defined as

$$\mathrm{Stab}_v G := G_v := \{g \in G \mid \underbrace{\rho(g)(v)}_{=:g\cdot v} = v\},$$

and the orbit of v is given by $G \cdot v := \{g \cdot v \mid g \in G\}$. The orbits form a decomposition of V and G_v is always a closed Lie subgroup of G. If v and $w = g \cdot v$ lie in the same orbit, the stabilizers G_v and G_w are conjugate, $G_w = g^{-1} \cdot G_v \cdot g$.

[4] We shall often identify a matrix Lie group with its image under its standard representation when no confusion is likely to occur.

1.2 Clifford algebras and spinors

Definition 1.2 *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, V a $\mathbb{K}$–vector space and suppose that f is a symmetric bilinear map on V. A pair (C, β) is called* Clifford algebra *of (V, f) if the following hold:*

1. *C is an associative $\mathbb{K}$–algebra with unit and $\beta : V \to C$ is a linear map.*
2. *$\beta(v) \cdot \beta(v) = f(v, v) \cdot 1$ for all $v \in V$.*
3. *If $u : V \to A$ is a linear map into an associative $\mathbb{K}$–algebra A with unit satisfying $u(v) \cdot u(v) = f(v, v) \cdot 1$ for all $v \in V$, then there is a unique algebra homomorphism $u' : C \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[rr, "\beta"] \arrow[dr, "u"'] & & C \arrow[dl, "u'"] \\ & A & \end{tikzcd}$$

For each (V, f), there is a up to isomorphism unique Clifford algebra, denoted by $Cl(V, f)$. We denote by $Cl(p, q)$ (or $Cl_{p,q}$) the Clifford algebra of $(\mathbb{R}^{p,q}, -\langle \cdot, \cdot \rangle_{p,q})$, where $\mathbb{R}^{p,q} \subset Cl(p, q)$ canonically. $Cl(p, q)$ is the unique associative real algebra with unit multiplicatively generated by the standard basis $(e_1, ..., e_n)$ of $\mathbb{R}^{p,q}$ with the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\langle e_i, e_j \rangle_{p,q} \cdot 1.$$

We denote the complexification of $Cl(p, q)$ by $Cl^{\mathbb{C}}(p, q)$, and this $\mathbb{C}$–algebra is isomorphic to the Clifford algebra $Cl(\mathbb{C}^n, -\langle \cdot, \cdot \rangle^{\mathbb{C}}_{p,q})$. Here, $\langle \cdot, \cdot \rangle^{\mathbb{C}}_{p,q}$ denotes the $\mathbb{C}$–bilinear extension of $\langle \cdot, \cdot \rangle_{p,q}$ to $\mathbb{C}^n \times \mathbb{C}^n$, where $n = p + q$. $Cl(p, q)$ contains the distinguished subgroups

- $Pin(p, q) := \{x_1 \cdot ... \cdot x_k \in Cl_{p,q} \mid x_j \in \mathbb{R}^{p,q}, \langle x_j, x_j \rangle_{p,q} = \pm 1\}$,
- $Spin(p, q) := \{x_1 \cdot ... \cdot x_{2l} \in Cl_{p,q} \mid x_j \in \mathbb{R}^{p,q}, \langle x_j, x_j \rangle_{p,q} = \pm 1\} \subset Cl^0(p, q)$,
- $Spin^+(p, q) := \{x_1 \cdot ... \cdot x_{2l} \in Cl_{p,q} \mid x_j \in \mathbb{R}^{p,q}, \langle x_j, x_j \rangle_{p,q} = \pm 1, |\{\langle x_j, x_j \rangle_{p,q} = +1\}| \text{ even }\}$.

These are Lie groups and $Spin^+(p, q)$ turns out to be the identity component of $Spin(p, q)$. Let $Spin^{(+)}(p, q)$ denote one of $Spin(p, q)$ and $Spin^+(p, q)$. Its Lie algebra $\mathfrak{spin}(p, q)$ can be identified with $Span\{e_k \cdot e_l \mid 1 \le k < l \le n\} \subset Cl(p, q)$, and the smooth group homomorphism

$$\lambda : Spin^{(+)}(p, q) \to SO^{(+)}(p, q),\ u \mapsto (\mathbb{R}^{p,q} \ni x \mapsto u \cdot x \cdot u^{-1} \in \mathbb{R}^{p,q}),$$

turns out to be a 2-fold covering map with differential given by $\lambda_*(e_k \cdot e_l) = 2E_{kl}$.

Theorem 1.3 ([Har90], Thm. 11.3) *As real associative algebras with unit, the Clifford algebras $Cl(p, q)$ are isomorphic to the following real matrix algebras:*

$q - p$ *mod* 8	$Cl(p, q) \cong$
0, 6	$M_N(\mathbb{R})$
2, 4	$M_N(\mathbb{H})$
1, 5	$M_N(\mathbb{C})$
3	$M_N(\mathbb{H}) \oplus M_N(\mathbb{H})$
7	$M_N(\mathbb{R}) \oplus M_N(\mathbb{R})$

The number N in each case can be easily computed by the fact that $\dim_{\mathbb{R}} Cl(p,q) = 2^n$. An explicit realisation of these isomorphisms is given in [DK06].

Theorem 1.4 ([Bau81]) *There are the following isomorphisms of complex algebras:*

n *mod* 2	$Cl^{\mathbb{C}}(p,q) \cong$
0	$M_N(\mathbb{C})$
1	$M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$

Remark 1.5 Let E, T, g_1 and g_2 denote the 2×2 matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \, g_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \, g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, let $\tau_j = \begin{cases} 1 & \epsilon_j = 1, \\ i & \epsilon_j = -1. \end{cases}$. Let $n = 2m$. In this case, $Cl^{\mathbb{C}}(p,q) \cong M_{2^m}(\mathbb{C})$ as complex algebras, and an explicit realisation of this isomorphism is given by

$$\Phi_{p,q}(e_{2j-1}) = \tau_{2j-1} \cdot E \otimes ... \otimes E \otimes g_1 \otimes \underbrace{T \otimes ... \otimes T}_{(j-1)\times},$$
$$\Phi_{p,q}(e_{2j}) = \tau_{2j} \cdot E \otimes ... \otimes E \otimes g_2 \otimes \underbrace{T \otimes ... \otimes T}_{(j-1)\times}.$$

Let $n = 2m+1$ and $q > 0$. In this case, there is an isomorphism $\widetilde{\Phi}_{p,q} : Cl^{\mathbb{C}}(p,q) \to M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$, given by

$$\widetilde{\Phi}_{p,q}(e_j) = (\Phi_{p,q-1}(e_j), \Phi_{p,q-1}(e_j)), \; j = 1, ..., 2m,$$
$$\widetilde{\Phi}_{p,q}(e_{2m+1}) = \tau_{2m+1}(iT \otimes ... \otimes T, -iT \otimes ... \otimes T).$$

Let in the following $Cl(V,f)$ denote one of the $\mathbb{K}$-algebras $Cl(p,q)$ or the complexificaion $Cl^{\mathbb{C}}(p,q) \cong Cl(\mathbb{C}^n, -\langle \cdot, \cdot \rangle^{\mathbb{C}}_{p,q})$. For $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $\mathbb{K} \subset K$, a K-representation of the $\mathbb{K}$-algebra $Cl(V,f)$ is a $\mathbb{K}$−algebra homomorphism

$$\rho : Cl(V,f) \to End_K(W),$$

where W is a finite-dimensional K-vector space[5] which is then called a $Cl(V,f)$-module. We write $\rho(\eta)(v) =: \eta \cdot v$ when no confusion is likely to occur. Theorem 1.3 and 1.4 directly lead to the next statement.

Theorem 1.6 ([LM89], Thm. 5.6 and 5.7) *Let $n = p + q$. If $q - p \not\equiv 3$ mod 4, there is (up to equivalence) exactly one irreducible real representation of $Cl(p,q)$. If $q - p \equiv 3$ mod 4, there are precisely two inequivalent real irreducible representations of $Cl(p,q)$. Furthermore, $Cl^{\mathbb{C}}(p,q)$ admits up to equivalence exactly one irreducible complex representation in case n is even and two such representations if n is odd.*

Thus, there is - up to equivalence - more than one real resp. complex irreducible representation of $Cl(V,f)$ exactly in cases $q - p \equiv 3$ mod 4 ($\mathbb{K} = \mathbb{R}$) or n odd ($\mathbb{K} = \mathbb{C}$), and

[5] or a right $\mathbb{H}$-module respectively.

in these cases the two inequivalent representations are distinguished as follows: Let $\mathbb{R}^{p,q}$ be equipped with the standard orientation and let $a_1, ..., a_n$ be any positively-oriented pseudo-orthonormal basis. Then the associated unit volume elements are defined to be

$$\omega_{\mathbb{R}} := a_1 \cdot .... \cdot a_n \in Cl(p,q),$$
$$\omega_{\mathbb{C}} := (-i)^{\lfloor \frac{n+1}{2} \rfloor + p} \omega_{\mathbb{R}} \in Cl^{\mathbb{C}}(p,q).$$

It holds that $\omega_{\mathbb{R}}^2 = (-1)^{\frac{n(n+1)}{2}+p}$ and $\omega_{\mathbb{C}}^2 = 1$. In particular, $\omega_{\mathbb{R}}^2 = 1$ iff $q-p = 0,3 \bmod 4$.

Proposition 1.7 ([LM89], Prop. 5.9) *Let $\rho : Cl(p,q) \to End_{\mathbb{R}}(W)$ be any irreducible real representation where $q - p \equiv 3 \bmod 4$. Then either $\rho(\omega_{\mathbb{R}}) = Id$ or $\rho(\omega_{\mathbb{R}}) = -Id$. Both possibilities can occur, and the resulting representations are inequivalent. The analogous statements are true in the complex case for $Cl^{\mathbb{C}}(p,q)$ and n odd.*

From now on, if there is more than one equivalence class of irreducible real resp. complex representations of $Cl(V,f)$, we shall *always choose one with* $\rho(\omega) = 1$. In our concrete realisation of the complex case from Remark 1.5, this corresponds to projection onto the first component. Having thus found a way to distinguish a up to equivalence unique real resp. complex irreducible representation for all Clifford algebras $Cl(V,f)$, we write

$$Cl(p,q) \to End_{\mathbb{R}}(\Delta^{\mathbb{R}}_{p,q}), \tag{1.3}$$
$$Cl^{\mathbb{C}}(p,q) \to End_{\mathbb{C}}(\Delta^{\mathbb{C}}_{p,q}) \tag{1.4}$$

for an irreducible real resp. complex representation of $Cl(p,q)$ resp. $Cl^{\mathbb{C}}(p,q)$. Considering the real and the complex case in common is done with the notation

$$Cl(V,f) \to End_{\mathbb{K}}(\Delta_{p,q}),$$

meaning (1.3), i.e. $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^n, f = -\langle\cdot,\cdot\rangle_{p,q}$ or (1.4), i.e. $\mathbb{K} = \mathbb{C}, V = \mathbb{C}^n, f = -\langle\cdot,\cdot\rangle^{\mathbb{C}}_{p,q}$. The previous discussion shows that the Clifford module $\Delta_{p,q}$ can be realised as vector space in either case as $\mathbb{K}^N$ for some N, where $\mathbb{K}$ is one of $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

Remark 1.8 In most cases one considers the standard indefinite scalar product $-x_1^2 - ... - x_p^2 + x_{p+1}^2 + ... + x_n^2$ on $\mathbb{R}^n$. However, we work with this more general notion of $\mathbb{R}^{p,q}$ only in order to simplify notation and upcoming calculations. The resulting representations of $Cl^{\mathbb{C}}_{p,q}$ are equivalent. More precisely, consider $\mathbb{R}^n$ with two different scalar products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ of signature (p,q) which both satisfy $\langle e_i, e_i\rangle_j = \pm 1$ for $i = 1,...,n,\ j = 1,2$. Let $f : (\mathbb{R}^n, \langle\cdot,\cdot\rangle_1) \to (\mathbb{R}^n, \langle\cdot,\cdot\rangle_2)$ be an orientation-preserving isometry such that

$$f(e_\alpha) = \pm e_\beta \ \forall \alpha, \beta \in \{1, ..., p+q\}$$

holds. By the universal property, f can be extended to an isomorphism $f : Cl^{\mathbb{C}}_{p,q,1} := Cl(\mathbb{R}^n, -\langle\cdot,\cdot\rangle_1)^{\mathbb{C}} \to Cl(\mathbb{R}^n, -\langle\cdot,\cdot\rangle_2)^{\mathbb{C}} =: C^{\mathbb{C}}_{p,q,1}$. Let Φ_i be an irreducible complex representation of $C^{\mathbb{C}}_{p,q,i}$ (with $\Phi_i(\omega_{\mathbb{C}}) = 1$ if n is odd) on a $\mathbb{C}$–vector space V. Then there is an isomorphism $\Phi : V \to V$ such that

$$\Phi_1(a)\,(\Phi(v)) = \Phi\,(\Phi_2(f(a))(v))$$

holds for all $a \in Cl_{p,q,1}$ and $v \in V$. For more details cf. [Kat99].

The complex spinor representation of $Spin^{(+)}(p,q)$ on $\Delta^{\mathbb{C}}_{p,q}$ is the homomorphism

$$\Phi : Spin^{(+)}(p,q) \to GL_{\mathbb{C}}(\Delta^{\mathbb{C}}_{p,q}),$$

obtained by restricting an irreducible complex representation $\Phi : Cl^{\mathbb{C}}(p,q) \to End_{\mathbb{C}}(\Delta^{\mathbb{C}}_{p,q})$ to $Spin^{(+)}(p,q) \subset Cl(p,q) \subset Cl^{\mathbb{C}}(p,q)$. The representation space $\Delta^{\mathbb{C}}_{p,q}$ is then referred to as complex spinor module. Its elements are called complex spinors. We use the same symbol $\Delta^{\mathbb{C}}_{p,q}$ for the $Spin^{(+)}(p,q)$- and $Cl^{\mathbb{C}}(p,q)$-module. Furthermore, when n is odd, the representation $\Delta^{\mathbb{C}}_{p,q}$ is irreducible, whereas in case n is even, $\Delta^{\mathbb{C}}_{p,q}$ turns out to be the direct sum of two inequivalent, irreducible complex representations of $Spin^{(+)}(p,q)$:

$$\Delta^{\mathbb{C}}_{p,q} = \Delta^{\mathbb{C},+}_{p,q} \oplus \Delta^{\mathbb{C},-}_{p,q}$$

The spaces $\Delta^{\mathbb{C},\pm}_{p,q}$ are exactly the ± 1 eigenspaces to the volume element involution $\omega_{\mathbb{C}}$ (cf. [Bau81]). Moreover, using our concrete realisation from Remark 1.5, one sees that $\Delta^{\mathbb{C}}_{p,q} \cong \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$, and a concrete basis we will work with is given by $u(\epsilon_1,...,\epsilon_m) := u(\epsilon_m) \otimes .... \otimes u(\epsilon_1)$, where $\epsilon_i = \pm 1$ and $u(1) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u(-1) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The real spinor representation of $Spin^{(+)}(p,q)$ on $\Delta^{\mathbb{R}}_{p,q}$ is the homomorphism

$$\rho : Spin^{(+)}(p,q) \to GL_{\mathbb{R}}(\Delta^{\mathbb{R}}_{p,q}),$$

obtained by restricting an irreducible real representation $\rho : Cl(p,q) \to End_{\mathbb{R}}(\Delta^{\mathbb{R}}_{p,q})$ (with $\rho(\omega_{\mathbb{R}}) = 1$ in case $q - p \equiv 3 \bmod 4$) to $Spin^{(+)}(p,q) \subset Cl(p,q)$. The representation space $\Delta^{\mathbb{R}}_{p,q}$ is referred to as real spinor module. Its elements are called real spinors.

Remark 1.9 We use the same symbol $\Delta^{\mathbb{R}}_{p,q}$ for the $Spin^{(+)}(p,q)$ and $Cl(p,q)$ module. If $q - p \equiv 0 \bmod 4$, the space $\Delta^{\mathbb{R}}_{p,q}$ is a reducible $Spin^{(+)}(p,q)$ module and it can be decomposed into the sum $\Delta^{\mathbb{R}}_{p,q} = \Delta^{\mathbb{R},+}_{p,q} \oplus \Delta^{\mathbb{R},-}_{p,q}$ of two irreducible, inequivalent $Spin^{(+)}(p,q)$-representations where in analogy to the complex case the spaces $\Delta^{\mathbb{R},\pm}_{p,q}$ are the ± 1 eigenspaces of the involution $\omega_{\mathbb{R}}$ (cf. [LM89], Prop. 5.10). Action by elements of $Pin(p,q)$ not being in $Spin(p,q)$ exchange these two summands. If $q - p \equiv 1, 2 \bmod 8$, the definition of $\Delta^{\mathbb{R}}_{p,q}$ turns out to be the sum of two equivalent real representations of $Spin^{(+)}(p,q)$. If $q - p \not\equiv 0 \bmod 4$ and $q - p \not\equiv 1, 2 \bmod 8$, the space $\Delta^{\mathbb{R}}_{p,q}$ is an irreducible $Spin^{(+)}(p,q)$-module. Interchanging p and q yields the same type of real spinor representation since $Spin(p,q) \cong Spin(q,p)$.

Having distinguished a -up to equivalence unique- real or complex irreducible representation $\chi : Cl(V,f) \to End(\Delta_{p,q})$, we define the Clifford multiplication of a vector $x \in \mathbb{R}^n$ by a spinor $\varphi \in \Delta_{p,q}$ to be

$$\begin{aligned} \mathbb{R}^n \times \Delta_{p,q} &\to \Delta_{p,q}, \\ (x,\varphi) &\mapsto x \cdot \varphi := cl(x)(\varphi) := \chi(x)(\varphi), \end{aligned}$$

which naturally extends to a multiplication by k-forms: For $\omega = \sum_{1 \le i_1 < ... < i_k \le n} \omega_{i_1...i_k} e^{\flat}_{i_1} \wedge ... \wedge e^{\flat}_{i_k} \in \Lambda^k_{p,q}$ and $\varphi \in \Delta_{p,q}$ we set

$$\omega \cdot \varphi := \sum_{1 \le i_1 < ... < i_k \le n} \omega_{i_1...i_k} e_{i_1} \cdot ... \cdot e_{i_k} \cdot \varphi \in \Delta_{p,q}. \tag{1.5}$$

Lemma 1.10 ([Fri00], section 1.5) *Given $g \in Spin(p,q), \omega \in \Lambda^k_{p,q}, \varphi \in \Delta_{p,q}$, it holds that*

$$g \cdot (\omega \cdot \varphi) = (\lambda(g)(\omega)) \cdot (g \cdot \varphi).$$

Here, we view $\lambda : Spin(p,q) \to SO(p,q) \subset GL(\mathbb{R}^n)$ as a $Spin(p,q)$ representation on $\mathbb{R}^n$ and then extend this map to a representation on $\Lambda^k_{p,q}$.

1.3 Structures on the space of spinors

Scalar products

We start with the definition of a scalar product on the spinor module in the complex case following [Bau81]. Our realisation of Clifford representation from Remark 1.4 yields that $\mathbb{C}^{2^m} = \Delta^{\mathbb{C}}_{p,q}$, where $n = 2m$ or $n = 2m+1$. Let $(v,w)_{\mathbb{C}} := \sum_{j=1}^{2^m} v_j \overline{w_j}$ denote the standard scalar product on $\mathbb{C}^{2^m}$ and introduce another bilinear form on this space by setting

$$\langle \varphi, \phi \rangle_{\Delta^{\mathbb{C}}_{p,q}} := i^{\frac{1}{2}p(p-1)} (e_1 \cdot \ldots \cdot e_p \cdot \varphi, \phi)_{\mathbb{C}} \tag{1.6}$$

for $\varphi, \phi \in \Delta^{\mathbb{C}}_{p,q}$. If $0 < p < n$, the map $\langle \cdot, \cdot \rangle_{\Delta^{\mathbb{C}}_{p,q}}$ is an indefinite Hermitian scalar product of signature $(2^{m-1}, 2^{m-1})$ on $\Delta^{\mathbb{C}}_{p,q}$. If $p \in \{0, n\}$, this scalar product is definite.

Proposition 1.11 ([KS12], section 2) *Let $n = p+q$ and fix an irreducible real representation $\rho : Cl(p,q) \to End_{\mathbb{R}}(\Delta^{\mathbb{R}}_{p,q}) \overset{\text{fix basis}}{\widehat{=}} M_N(\mathbb{R})$, where $N = dim_{\mathbb{R}} \Delta^{\mathbb{R}}_{p,q}$. Furthermore, let $\langle \cdot, \cdot \rangle$ be a real-valued bilinear form on $\Delta^{\mathbb{R}}_{p,q}$ such that vectors are* ***self-adjoint up to sign****, that is there is an overall sign $\pm$ such that*

$$\langle x \cdot v, w \rangle = \pm \langle v, x \cdot w \rangle$$

for all $x \in \mathbb{R}^n$ and $v, w \in \Delta^{\mathbb{R}}_{p,q}$. Then the Gram matrix of $\langle \cdot, \cdot \rangle$ is with respect to the fixed basis and up to scale one of $\rho(e_1 \cdot \ldots \cdot e_p)$ or $\rho(e_{p+1} \cdot \ldots \cdot e_n)$.

In the following, we choose the bilinear form coming from $e_1 \cdot \ldots \cdot e_p$. We denote the resulting inner product on $\Delta^{\mathbb{R}}_{p,q}$ by $\langle \cdot, \cdot \rangle_{\Delta^{\mathbb{R}}_{p,q}}$, i.e. we define in the setting of the last Proposition, where $\Delta^{\mathbb{R}}_{p,q} \cong \mathbb{R}^N$ for some N (as vector spaces) and where $(\cdot, \cdot)$ denotes the standard Euclidean scalar product on $\mathbb{R}^N$

$$\langle v, w \rangle_{\Delta^{\mathbb{R}}_{p,q}} := (e_1 \cdot \ldots \cdot e_p \cdot v, w). \tag{1.7}$$

As $e_1 \cdot \ldots \cdot e_p \in Cl(p,q)$ is invertible, this bilinear form is nondegenerate. One checks that (cf. [KS12]) $\langle \cdot, \cdot \rangle_{\Delta^{\mathbb{R}}_{p,q}}$ is symmetric if $p = 0, 1 \bmod 4$ with neutral signature ($p \neq 0$ and $q \neq 0$) or it is definite ($p = 0$ or $q = 0$). In case $p = 2, 3 \bmod 4$, this bilinear form is anti-symmetric, and thus, $(\Delta^{\mathbb{R}}_{p,q}, \langle \cdot, \cdot \rangle_{\Delta^{\mathbb{R}}_{p,q}})$ is a symplectic vector space.

Remark 1.12 One word about notation: As it turns out, the so defined real and complex inner products (note that we fixed representations to define them) share some important properties. Therefore, it might be useful to handle both of them with one common symbol. So we let $\langle \cdot, \cdot \rangle_{\Delta_{p,q}}$ denote $\langle \cdot, \cdot \rangle_{\Delta^{\mathbb{R},\mathbb{C}}_{p,q}}$ on $\Delta^{\mathbb{R},\mathbb{C}}_{p,q}$ and also write

$$\langle v, w \rangle_{\Delta_{p,q}} = c_1(p)(e_1 \ldots e_p v, w), \tag{1.8}$$

where the representations are fixed as described above[6], $(\cdot,\cdot)$ is the standard scalar product on $\mathbb{K}^N \cong \Delta^{\mathbb{K}}_{p,q}$ and $c_1(p) := \begin{cases} i^{\frac{1}{2}p(p-1)} & \mathbb{K} = \mathbb{C}, \\ 1 & \mathbb{K} = \mathbb{R}. \end{cases}$.

Lemma 1.13 (cf. [Bau81]) *The inner products $\langle\cdot,\cdot\rangle_{\Delta_{p,q}}$ on $\Delta_{p,q}$ satisfy:*

1. *For all $x \in \mathbb{R}^{p,q}$ and $v, w \in \Delta_{p,q}$ it holds that*

$$\langle x \cdot v, w\rangle_{\Delta_{p,q}} + (-1)^p \langle v, x \cdot w\rangle_{\Delta_{p,q}} = 0.$$

In particular, $\langle\cdot,\cdot\rangle_{\Delta_{p,q}}$ is $Spin^+(p,q)$-invariant, i.e $\langle g\cdot v, g\cdot w\rangle_{\Delta_{p,q}} = \langle v, w\rangle_{\Delta_{p,q}}$ for all $v, w \in \Delta_{p,q}$ and $g \in Spin^+(p,q)$.

2. *If p, q are even, then $\Delta^{\pm}_{p,q}$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\Delta_{p,q}}$, and if p, q are odd, then $\Delta^{\pm}_{p,q}$ are totally lightlike with respect to $\langle\cdot,\cdot\rangle_{\Delta_{p,q}}$ (in the real case this of course only makes sense in signatures $q - p \equiv 0 \mod 4$).*

3. *Using the explicit realisation of Clifford multiplication from Remark 1.5 with the inner product $\langle\cdot,\cdot\rangle_{p,q}$ on $\mathbb{R}^{p,q}$ given by $\epsilon_i = (-i)^j$ for $1 \le i \le 2p$ and $\epsilon_i = 1$ for $i > 2p$ we have in these cases that*

$$\langle u(\epsilon_1,...,\epsilon_m), u(\delta_1,...,\delta_m)\rangle_{\Delta^{\mathbb{C}}_{p,q}} \neq 0 \text{ iff } (\epsilon_1,...,\epsilon_p,\epsilon_{p+1},...,\epsilon_m) = (-\delta_1,...,-\delta_p,\delta_{p+1},...,\delta_m), \tag{1.9}$$

and in case that this scalar product is nonzero, it equals some power of i not depending on $\epsilon_{i>p}$.

Remark 1.14 If one does not want to fix a certain representation in order to introduce $\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{R}}_{p,q}}$, one can proceed as in [AC97]: We call a bilinear form β on $\Delta^{\mathbb{R}}_{p,q}$ admissible if it is $Spin^+(p,q)$-invariant, β is symmetric or skew-symmetric, Clifford multiplication is either β–symmetric or β–skew symmetric, and if $\Delta^{\mathbb{R}}_{p,q} = \Delta^{\mathbb{R},+}_{p,q} \oplus \Delta^{\mathbb{R},-}_{p,q}$ is reducible, then $\Delta^{\mathbb{R},\pm}_{p,q}$ are either mutually orthogonal or isotropic. The above discussion yields that there exists always one admissible inner product on $\Delta^{\mathbb{R}}_{p,q}$, and in fact we could as well replace in the following $\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{R}}_{p,q}}$ by any admissible inner product, which are classified in [AC97].

Real and Quaternionic Structures on $\Delta^{\mathbb{C}}_{p,q}$

Theorem 1.3 makes clear that depending on the number $q-p \mod 8$ the restriction of suitable irreducible representations of the *real* Clifford algebra $Cl(p,q)$ leads to complex, real or quaternionic representations of $Spin(p,q)$. In physics literature, this leads to the notion of Majorana and symplectic Majorana spinors, cf. [Far05]. On the other hand, following [LM89] we get a more uniform treatment of these spinor representations if we consider all irreducible representations of $Cl(p,q)$ as in Theorem 1.3 as being real. This will be our standpoint mostly throughout the text.

[6]Note that one has to fix certain irreducible representations in order to introduce $\langle\cdot,\cdot\rangle_\Delta$. If one fixes a different irreducible complex representation of $Cl^{\mathbb{C}}(p,q)$ on $\Delta^{\mathbb{C}}_{p,q} \cong \mathbb{C}^N$, the definition and the properties of the scalar product are analogous but the factor $c_1(p)$, which ensures that the product is Hermitian, might change.

Depending on the number $q-p$ mod 8, the spinor modules $\Delta^{\mathbb{R}}_{p,q}$ and $\Delta^{\mathbb{C}}_{p,q}$ are related by the existence of real or quaternionic structures[7] on $\Delta^{\mathbb{C}}_{p,q}$ which commute or anticommute with Clifford multiplication, in particular they are *Spin*–equivariant. This theory of real and quaternionic structures on the spinor module can be found elsewhere in whole detail, see [Har90, LM89, Fri00].

Remark 1.15 (At least) In signatures (p,q) with $q-p\equiv 0,7$ mod 8 (in particular, this includes the split signatures (p,p) and $(p+1,p)$) one obtains a uniform treatment of real and complex spinor representations in the sense that $\Delta^{\mathbb{C}}_{p,q}=\Delta^{\mathbb{R}}_{p,q}\otimes_{\mathbb{R}}\mathbb{C}$. The link between $\Delta^{\mathbb{C}}_{p,q}$ and $\Delta^{\mathbb{R}}_{p,q}$ is given by the existence of real structures $\alpha:\Delta^{\mathbb{C}}_{p,q}\to\Delta^{\mathbb{C}}_{p,q}$ commuting with Clifford multiplication. In this picture, $\Delta^{\mathbb{R}}_{p,q}=Eig(\alpha,+1)\subset\Delta^{\mathbb{C}}_{p,q}$.
One can for these signatures then also compare the scalar products constructed on $\Delta^{\mathbb{R}}_{p,q}$ and $\Delta^{\mathbb{C}}_{p,q}$. Fix a real Clifford module $\rho:Cl(p,q)\to End_{\mathbb{R}}(\Delta^{\mathbb{R}}_{p,q})\subset M_N(\mathbb{R})$ and extend ρ to a complex representation of $Cl^{\mathbb{C}}(p,q)$ on $\Delta^{\mathbb{C}}_{p,q}=\mathbb{C}^N$ by complexification. Now form $\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{C},\mathbb{R}}_{p,q}}$ using these two representations. Then our construction yields

$$\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{R}}_{p,q}}=c\cdot\left(\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{C}}_{p,q}}\right)_{|\Delta^{\mathbb{R}}_{p,q}\times\Delta^{\mathbb{R}}_{p,q}},$$

where $c\in\mathbb{C}\backslash\{0\}$ is a constant.

Decomposition of $\Delta_{p+1,q+1}$

There is an important decomposition of $\Delta_{p+1,q+1}$ into $Spin(p,q)$–modules. Let $(e_-,...,e_+)$ denote the basis of $\mathbb{R}^{p+1,q+1}$ with lightlike directions $e_\pm:=\frac{1}{\sqrt{2}}(e_{n+1}\pm e_0)$ as introduced in Remark 1.1. One then has a decomposition $\mathbb{R}^{p+1,q+1}=\mathbb{R}e_-\oplus\mathbb{R}^{p,q}\oplus\mathbb{R}e_+$ of $\mathbb{R}^{p+1,q+1}$ into irreducible $O(p,q)$–modules. We define the annihilation spaces $Ann(e_\pm):=\{v\in\Delta_{p+1,q+1}\mid e_\pm\cdot v=0\}$. For every $v\in\Delta_{p+1,q+1}$ there is a unique $w\in\Delta_{p+1,q+1}$ such that $v=e_-\cdot w+e_+\cdot w\in Ann(e_-)\oplus Ann(e_+)$, leading to a decomposition

$$\Delta_{p+1,q+1}=Ann(e_-)\oplus Ann(e_+) \tag{1.10}$$

with corresponding projections $proj_{Ann(e_\pm)}:\Delta_{p+1,q+1}\to Ann(e_\pm)$. As $x\cdot e_\pm=-e_\pm\cdot x$ for all $x\in\mathbb{R}^{p,q}\cong\text{span}(e_1,...,e_n)\subset\mathbb{R}^{p+1,q+1}$, we see that $\mathbb{R}^{p,q}\subset\mathbb{R}^{p+1,q+1}$ and $Spin(p,q)\subset Spin(p+1,q+1)$ act on $Ann(e_\pm)$. One can conclude that $\Delta_{p,q}$ is isomorphic to $Ann(e_\pm)$ as $Spin(p,q)$ representation space. In order to make this relation precise, we fix an isomorphism $\chi:Ann(e_-)\to\Delta_{p,q}$ of $Spin(p,q)$-representations. Then there is an induced isomorphism $\zeta:Ann(e_+)\to\Delta_{p,q}$, $v\mapsto\chi(e_-v)$, and an isomorphism

$$\begin{aligned}\Pi:\Delta_{p+1,q+1|Spin(p,q)}&\cong\Delta_{p,q}\oplus\Delta_{p,q},\\ v=e_+w+e_-w&\mapsto\begin{pmatrix}\chi(e_-e_+w)\\ \chi(e_-w)\end{pmatrix}\end{aligned} \tag{1.11}$$

[7] Recall that for V be a complex vector space, a real structure on V is an $\mathbb{R}$-linear map $\alpha:V\to V$ with the properties $\alpha^2=Id$, $\alpha(iv)=-i\alpha(v)$ and a quaternionic structure on V is an $\mathbb{R}$-linear map $\beta:V\to V$ such that $\beta^2=-Id$, $\beta(iv)=-i\beta(v)$. Given a real structure $\alpha:V\to V$, the vector space V can be decomposed into $V=V_{\mathbb{R}}\oplus iV_{\mathbb{R}}=Eig(\alpha,1)\oplus iEig(\alpha,1)$ according to the ±1 eigenspaces of α.

of $Spin(p,q)$-modules. One calculates (cf. [HS11a]) that wrt. this decomposition the scalar product $\langle\cdot,\cdot\rangle_{\Delta_{p+1,q+1}}$ is given by

$$\langle\begin{pmatrix}v_1\\w_1\end{pmatrix},\begin{pmatrix}v_2\\w_2\end{pmatrix}\rangle_{\Delta_{p+1,q+1}} = -\frac{\delta^p}{\sqrt{2}}\left(\langle v_1,w_2\rangle_{\Delta_{p,q}} + (-1)^p\langle w_1,v_2\rangle_{\Delta_{p,q}}\right) \tag{1.12}$$

where $v_j, w_j \in \Delta_{p,q}$ for $j = 1,2$. The factor $\delta \in \mathbb{K}$ depends on the chosen admissible spin scalar product only.

Distinguished orbits in $\Delta_{p,q}$

Given an irreducible complex representation $\Phi : Cl^{\mathbb{C}}(p,q) \to End_{\mathbb{C}}(\Delta^{\mathbb{C}}_{p,q})$ (satisfying $\Phi(\omega_{\mathbb{C}}) = 1$ in case n odd), the Clifford multiplication $\mathbb{R}^n \times \Delta^{\mathbb{C}}_{p,q} \to \Delta^{\mathbb{C}}_{p,q}$ can be extended to a complex bilinear map $\mathbb{C}^n \times \Delta^{\mathbb{C}}_{p,q} \to \Delta^{\mathbb{C}}_{p,q}$ (by restriction of Φ to $\mathbb{C}^n$). Now one associates to every spinor $v \in \Delta^{\mathbb{C}}_{p,q}$ the subspaces

$$\ker_{\mathbb{C}} v := \{X \in \mathbb{C}^n \mid X\cdot v = 0\} \text{ and } \ker v := \{X \in \mathbb{R}^n \mid X \cdot v = 0\}.$$

$\ker_{\mathbb{C}} v$ is isotropic with respect to the complex linear extension $\langle\cdot,\cdot\rangle^{\mathbb{C}}_{p,q}$ of $\langle\cdot,\cdot\rangle_{p,q}$, and in particular, $\ker v$ is isotropic with respect to $\langle\cdot,\cdot\rangle_{p,q}$. Consider a spinor $v \in \Delta_{p,q}$ and its associated totally lightlike subspace $\ker v \subset \mathbb{R}^{p,q}$. Clearly the dimension of this space depends on the $Spin(p,q)$-orbit of v only. In case $\ker v \neq \{0\}$ we say that v has positive nullity.

Proposition 1.16 ([TT94]) *The set of all spinors of positive nullity is contained in the set of all spinors $w \in \Delta_{p,q}$ such that 0 is in the closure of the orbit $Spin^+(p,q)\cdot w$.*

This has an immediate consequence: Let us call a continuous function $J : \Delta_{p,q} \to \mathbb{K}$ an invariant of the $Spin^+(p,q)$-action if $J(g\cdot v) = J(v)$ for all $g \in Spin^+(p,q)$ and $v \in \Delta_{p,q}$. If v is a spinor of positive nullity and J is an invariant then clearly $J(v) = J(0)$. In particular, $\langle v,v\rangle_{\Delta_{p,q}} = 0$ for every spinor of positive nullity as $\langle\cdot,\cdot\rangle_{\Delta_{p,q}}$ is an invariant. We now consider the extremal case of *maximal* nullity, following [Kat99].

Definition 1.17 *A complex spinor $v \in \Delta^{\mathbb{C}}_{p,q}$ is said to be pure if $\dim_{\mathbb{C}} \ker_{\mathbb{C}} v = \lfloor \frac{n}{2} \rfloor$, i.e. if its kernel under (extended) Clifford multiplication is a maximally isotropic subspace. In the split signatures (m,m) and $(m+1,m)$ we call $v \in \Delta^{\mathbb{R}}_{p,q}$ (real) pure if $\dim_{\mathbb{R}} \ker v = m$, i.e. $\ker v$ is of maximal dimension.*

So in the following, when talking about pure spinors, we mean either the complex case or real pure spinors in split signature. Using explicit realisations of spinor representations, one easily shows that there always exist pure spinors. If $n = 2m$, pure spinors are contained in $\Delta^+_{p,q} \cup \Delta^-_{p,q}$. Moreover, let $v, w \in \Delta_{p,q}$ be pure spinors, then

$$\langle v,w\rangle_{\Delta_{p,q}} = 0 \Leftrightarrow \ker v \cap \ker w \neq \{0\}. \tag{1.13}$$

Remark 1.18 [Bry00] discusses the orbit structure of real pure spinors under the $Spin^+(p,q)$-action in more detail. Let $(p,q) = (m+1,m)$. In this case, the space of pure spinors forms a single orbit in $\Delta^{\mathbb{R}}_{p,q}$ which is a cone and has dimension $\frac{1}{2}m(m+1)+1$, being the lowest dimension for a nonzero orbit. In case $(p,q) = (m,m)$, the space of pure spinors consists of precisely two orbits, contained in $\Delta^{\pm}_{p,q}$ respectively. The dimension of each of these orbits is $\frac{1}{2}m(m-1)+1$, and again, this is the minimal dimension for a nonzero orbit.

By definition, real pure spinors are in tight relationship to nullplanes in $\mathbb{R}^{p,q}$. We call an $m-$dimensional (maximally) isotropic subspace in $\mathbb{R}^{m,m}$ a m-nullplane.

Proposition 1.19 ([Har90], Thm. 12.100) *Let $\Delta^{\mathbb{R},pure}_{m,m}$ denote the set of all real pure spinors in signature (m,m) and let N be the space of all $m-$nullplanes in $\mathbb{R}^{m,m}$. The map*

$$\Delta^{\mathbb{R},pure}_{m,m} \ni v \mapsto ker\ v \in N$$

factors to a bijection $\Delta^{\mathbb{R},pure}_{m,m}/\mathbb{R}^* \cong N$.

Remark 1.20 One obtains analogous results in the real split case $(m+1,m)$ and also similar statements can be made in the complex case using the theory of projective (pure) spinors (cf. [Kat99]). Further, note that some authors also call real spinors $v \in \Delta^{\mathbb{R}}_{p,q}$ in other signatures pure if ker v is maximally isotropic, i.e. dim ker $v = \min\{p,q\}$. It is clear from the representation theory of Clifford algebras that these objects always exist. However, they miss important properties such as the simple orbit structure or their relation to maximal nullplanes from Proposition 6.20.

[Kat99] computes the stabilizer of a real pure spinor with respect to the $Spin^+(p,q)$ action on $\Delta^{\mathbb{R}}_{p,q}$ in the split case. We only list results for the case $(m+1,m)$. One obtains results for signatures (m,m) by omitting the last basis vector, last coordinate etc. We work with the split signature scalar product $\langle\cdot,\cdot\rangle_{split}$ on $\mathbb{R}^{m+1,m}$ satisfying that $\epsilon_i = (-1)^i$ and introduce the basis $(f_1^+,...f_m^+,f_1^-,...,f_m^-,e_{2m+1})$, where

$$f_i^+ := e_{2i-1} + e_{2i}\ ,\ f_i^- := e_{2i-1} - e_{2i} \text{ for } 1 \le i \le m.$$

The following $n\times n$-matrices are all with respect to this basis. Let $v \in \Delta^{\mathbb{R}}_{m+1,m}$ be a real pure spinor and let $Stab_v \subset Spin(m+1,m)$ denote the isotropy group of v with respect to the $Spin(m+1,m)$ action on $\Delta^{\mathbb{R}}_{m+1,m}$. As $-1 \notin Stab_v$, the double covering map λ restricts to an isomorphism $\lambda_{|Stab_v} : Stab_v \to \lambda(Stab_v) =: H_v$ and we denote $H_v^+ := H_v \cap SO^+(m+1,m)$.

Lemma 1.21 *In the above setting, it holds (up to conjugation in $SO^+(m+1,m)$) that*

$$\begin{aligned} H_v &\cong R(m+1,m) := \left\{ h(A,B,X,Y) := \begin{pmatrix} A & B & X \\ 0 & \left(A^T\right)^{-1} & 0 \\ 0 & Y & 1 \end{pmatrix} \;\middle|\; \begin{matrix} 2B^T(A^T)^{-1} + 2A^{-1}B + Y^TY = 0 \\ 2A^{-1}X + Y^T = 0 \\ det\ A = \pm 1 \end{matrix} \right\}, \\ H_v^+ &\cong R^+(m+1,m) := \{\, h(A,B,X,Y) \in H_v \mid det\ A = 1\}. \end{aligned} \tag{1.14}$$

$R^+(m+1,m)$ is a semidirect product $R^+(m+1,m) = SL(m,\mathbb{R}) \ltimes N$ of $SL(m,\mathbb{R})$ and the nilpotent group N given by

$$N = \left\{ \begin{pmatrix} I_m & B & X \\ 0 & I_m & 0 \\ 0 & -2X^T & 1 \end{pmatrix} \;\middle|\; B \in M_m(\mathbb{R}), B^T + B + 2XX^T = 0, X \in \mathbb{R}^m \right\}.$$

1.4 Associated forms to a spinor

In the Lorentzian case one can associate to every nonzero spinor a nonvanishing vector, the so called Dirac current (cf. [BL04, Lei01]). Following [AC97] we generalize this construction as follows: For $n = r + s$ we fix $\Delta_{r,s}$[8] together with an admissible inner product $\langle \cdot, \cdot \rangle_{\Delta_{r,s}}$. We then define

$$\begin{aligned} \Gamma^k : \Delta_{r,s} \times \Delta_{r,s} \to \Lambda^k_{r,s},\ (\chi_1, \chi_2) \mapsto \alpha^k_{\chi_1,\chi_2},\ \text{where} \\ \langle \alpha^k_{\chi_1,\chi_2}, \alpha \rangle_{r,s} := d_{k,r} \left(\langle \alpha \cdot \chi_1, \chi_2 \rangle_{\Delta_{r,s}} \right)\ \forall \alpha \in \Lambda^k_{r,s}. \end{aligned} \tag{1.15}$$

The map $d_{k,r} : \mathbb{K} \to \mathbb{K}$ is the identity for $\mathbb{K} = \mathbb{R}$, whereas for $\mathbb{K} = \mathbb{C}$ it is defined as follows: One finds for complex spinors $\chi \in \Delta^{\mathbb{C}}_{p,q}$ that $\langle \alpha \cdot \chi, \chi \rangle_{\Delta^{\mathbb{C}}_{r,s}}$ is either real or purely imaginary. This depends on (r, s) and k as well as the chosen representation and admissible scalar product, but not on χ. One then chooses $d_{k,r} \in \{Re, Im\}$ so that $\alpha_\chi := \alpha^k_{\chi,\chi}$is indeed a real form and -if possible- nontrivial. It is obvious that the algebraic Dirac form α^k_χ is explicitly given by the formula

$$\alpha^k_\chi = \sum_{1 \le i_1 < i_2 < \ldots < i_l \le n} \epsilon_{i_1} \ldots \epsilon_{i_l} \cdot d_{k,r} \left(\langle e_{i_1} \cdot \ldots e_{i_l} \cdot \chi, \chi \rangle_{\Delta_{r,s}} \right) e^\flat_{i_1} \wedge \ldots \wedge e^\flat_{i_l}. \tag{1.16}$$

For $k = 1$ the vector $V_\chi := \left(\alpha^1_\chi \right)^\flat$ appears under the name Dirac current in physics literature. The construction is nontrivial at least for $k = r$ since $\alpha^r_\chi = 0 \Leftrightarrow \chi = 0$.

Proposition 1.22 *For $\chi \in \Delta_{r,s}$, $k \in \mathbb{N}$ and $g \in Spin^+(p, q)$ one has the equivariance*

$$\alpha^k_{g \cdot \chi} = \lambda(g)(\alpha^k_\chi).$$

Proof. Using Lemma 1.10 and the $Spin^+(r, s)$ invariance of $\langle \cdot, \cdot \rangle_{\Delta_{r,s}}$, we obtain

$$\begin{aligned} \langle \alpha^k_{g \cdot \chi}, \alpha \rangle_{r,s} &= d_{k,r} \left(\langle \alpha \cdot (g \cdot \chi), g \cdot \chi \rangle_{\Delta_{r,s}} \right) = d_{k,r} \left(\langle g^{-1} \cdot (\alpha \cdot (g \cdot \chi)), \chi \rangle_{\Delta_{r,s}} \right) \\ &= d_{k,r} \left(\langle (\lambda(g^{-1})\alpha) \cdot g^{-1} \cdot (g \cdot \chi), \chi \rangle_{\Delta_{r,s}} \right) = \langle \alpha^k_\chi, \lambda(g^{-1})\alpha \rangle_{r,s} = \langle \lambda(g)(\alpha^k_\chi), \alpha \rangle_{r,s} \end{aligned}$$

for all forms $\alpha \in \Lambda^k_{r,s}$, and the assertion follows from the definition of $\alpha^k_{g \cdot \chi}$. □

Remark 1.23 In the real case, i.e. considering $\Delta^{\mathbb{R}}_{r,s}$ with admissible inner product, the map Γ^k is always totally symmetric or antisymmetric in its arguments, depending on k and r: We say that $\sigma(\Gamma^k) = 1$ if Γ^k is symmetric and -1 if it is antisymmetric. It holds that (cf. [AC08])

$$\sigma(\Gamma^k) = (-1)^{r(r-1) + k \cdot (r+1) + \frac{k(k-1)}{2}}. \tag{1.17}$$

There is an important relation between the structure of α^r_χ and ker χ, which relates certain algebraic Dirac forms to distinguished orbits in $\Delta_{r,s}$ discussed above.

Lemma 1.24 *Let $\chi \in \Delta_{p,q} \backslash \{0\}$ and let $k := dim\ ker\ \chi (\le p)$. Then α^p_χ can be written as*

$$\alpha^p_\chi = l^\flat_1 \wedge \ldots \wedge l^\flat_k \wedge \widetilde{\alpha}, \tag{1.18}$$

[8] In this section we change the notation from (p, q) to (r, s) because we will later apply these results in cases in conformal geometry, where $(r, s) = (p, q)$ and $(r, s) = (p + 1, q + 1)$.

where $l_j \in \mathbb{R}^{p,q}$ for $1 \le j \le k$ such that span $\{l_1, ..., l_k\} = ker\ \chi$ (in particular, this implies that the l_j are lightlike and mutually orthogonal), $\widetilde{\alpha} \in \Lambda^{p-k}\left(\left(ker\ \chi\right)^{\perp}\right)^*$ and (1.18) is maximal in the sense that there exists no lightlike vector l_{k+1} being orthogonal to l_i for $1 \le i \le k$ such that $\alpha^p_\chi = l_1^\flat \wedge ... \wedge l_k^\flat \wedge l_{k+1}^\flat \wedge \widetilde{\widetilde{\alpha}}$. Moreover, whenever α^p_χ can be written as in (1.18) for mutually orthogonal lightlike vectors $l_1, ..., l_k$, it follows that $l_1, ..., l_k \in ker\ \chi$.

Proof. We may assume that the scalar product on $\mathbb{R}^{p,q}$ is chosen such that $\epsilon_i = (-1)^j$ for $j = 1, ..., 2p$ and $\epsilon_i = 1$ for $j > 2p$ (cf. Remark 1.8). We now fix the complex representation of $Cl^{\mathbb{C}}_{p,q}$ from Remark 1.5, introduce the lightlike directions

$$f_i^+ := e_{2i-1} + e_{2i},\ f_i^- := e_{2i-1} - e_{2i},\ 1 \le i \le p \tag{1.19}$$

and calculate

$$\Phi_{p,q}(f_i^+) = E \otimes ... \otimes E \otimes \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \otimes \underbrace{T \otimes ... \otimes T}_{(i-1)\times},$$

$$\Phi_{p,q}(f_i^-) = E \otimes ... \otimes E \otimes \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \otimes \underbrace{T \otimes ... \otimes T}_{(i-1)\times}.$$

If the spinor χ is given and $H := \ker \chi$, we can always find $g \in Spin^+(p,q)$ and $k \in \mathbb{N}$ such that $\lambda(g)(H) = \text{span}(f_1^{\delta_1}, ..., f_k^{\delta_k})$ for some $\delta_j \in \{\pm 1\}$ and $j = 1, ..., k$, and therefore, by using the equivariance property from Proposition 1.22, we may assume that $H = \text{span}(f_1^{\delta_1}, ..., f_k^{\delta_k})$ for some $\delta_j = \pm 1$. With respect to the unitary basis $u(\epsilon)$ of $\Delta_{p,q}$, the spinor χ can be represented as

$$\chi = \sum_{(\nu_1,...,\nu_m)\in\{\pm 1\}^m} a_{\nu_1,...,\nu_m} \cdot u(\nu_1, ..., \nu_m),\ \text{with } a_{\nu_1,...,\nu_m} \in \mathbb{C}. \tag{1.20}$$

We have

$$f_i^{\delta_i} \cdot u(\nu_1, ..., \nu_m) = \begin{cases} 0 & \delta_i = \nu_i \\ (\pm 2) \cdot u(\nu_1, ..., -\nu_i, ..., \nu_m) & \delta_i = -\nu_i \end{cases}$$

and consequently,

$$\chi = \sum_{(\nu_{k+1},...,\nu_m)\in\{\pm 1\}^{m-k}} a_{\nu_{k+1},...,\nu_m} \cdot u(\delta_1, ..., \delta_k, \nu_{k+1}, ..., \nu_m). \tag{1.21}$$

We now work with the basis $(f_1^+, f_1^-, ..., f_p^+, f_p^-, e_{2p+1}, ..., e_n)$ of $\mathbb{R}^{p,q}$. Let $(b_1, ..., b_p)$ be a $p-$tuple of ordered basis elements. Using the scalar product formula (1.9) it follows that

$$\langle b_1 \cdot ... \cdot b_p \cdot \chi, \chi \rangle_{\Delta_{p,q}} = 0,\ \text{unless } b_j = f_j^{-\delta_j} \text{ for } j = 1, ..., k.$$

But then it is a direct consequence of the definition of α^p_χ and the property that $\langle f_i^{-\delta_i}, b_j \rangle \neq 0$ iff $b_j = f_i^{\delta_i}$ that $\alpha^p_\chi = \left(f_1^{\delta_1}\right)^\flat \wedge ... \wedge \left(f_k^{\delta_k}\right)^\flat \wedge \widetilde{\alpha}$ for some $p-k$-form $\widetilde{\alpha}$ which lives on $H^\perp$. Conversely, suppose that for a given spinor χ its associated Dirac-p-form α^p_χ can be expressed as in (1.18). Again, by making use of the equivariance property, we may assume that $\alpha^p_\chi = \left(f_1^{\delta_1}\right)^\flat \wedge ... \wedge \left(f_k^{\delta_k}\right)^\flat \wedge \widetilde{\alpha}$. We have to show that $f_i^{\delta_i} \cdot \chi = 0$. The definition of α^p_χ yields that for all $\eta \in \{\pm 1\}^p$ one has that

$$\langle f_1^{\eta_1} \cdot .... \cdot f_p^{\eta_p} \cdot \chi, \chi \rangle_{\Delta_{p,q}} = 0 \text{ unless } \eta_j = -\delta_j \text{ for } j = 1, ..., k. \tag{1.22}$$

We write χ as in (1.20) and let $(\eta_1, ..., \eta_k) \neq -(\delta_1, ..., \delta_k)$. Then (1.22) translates into

$$0 = \sum_{\nu,\mu \in \{\pm 1\}^m} a_\nu \overline{a_\mu} \cdot \langle f_1^{\eta_1} \cdot ... \cdot f_p^{\eta_p} u(\nu_1, ..., \nu_m), u(\mu_1, ..., \mu_m)\rangle_{\Delta_{p,q}}.$$

However, $\langle f_1^{\eta_1} \cdot ... \cdot f_p^{\eta_p} \cdot u(\nu_1, ..., \nu_m), u(\mu_1, ..., \mu_m)\rangle_{\Delta_{p,q}} \neq 0$ iff $\nu = \mu$ and $(\mu_1, ..., \mu_p) = (-\eta_1, ..., -\eta_p)$ as follows from the scalar product formula (1.9), and in these cases the value of the scalar product does not depend on the index tuple ν. Consequently,

$$0 = \sum_{(\nu_{p+1},...,\nu_m)} |a_{-\eta_1,...,-\eta_p,\nu_{p+1},...,\nu_m}|^2.$$

That is, χ can be expressed as in (1.21) and $f_i^{\delta_i} \cdot \chi = 0$ for $i = 1, ..., k$. These two observations prove the Proposition. □

Remark 1.25 The previous two statements generalize well-known facts about the associated Dirac current V_χ to a spinor $\chi \in \Delta_{1,n-1}$ in the Lorentzian case from [Lei01]: It holds that $\|V_\chi\|^2 = 0$ implies that $V_\chi \cdot \chi = 0$ being is a special case of Lemma 1.24, and V_χ is always causal. Moreover, all possible algebraic Dirac forms α^2_φ for $0 \neq \varphi \in \Delta^{\mathbb{C}}_{2,n-2}$ have been classified in [Lei07]. Precisely one of the following cases occurs:

1. $\alpha^2_\varphi = l_1^\flat \wedge l_2^\flat$, where l_1, l_2 span a totally lightlike plane in $\mathbb{R}^{2,n-2}$.
2. $\alpha^2_\varphi = l^\flat \wedge t^\flat$ where l is lightlike, t is a orthogonal timelike vector.
3. $\alpha^2_\varphi = \omega$ (up to conjugation in $SO(2, n-2)$), where $n = 2m$ is even and ω is equivalent to the standard Kähler form ω_0 [9] on $\mathbb{R}^{2,n-2}$. In this case, $Stab_{\alpha^2_\varphi} O(2, n-2) \subset U(1, m-1)$.
4. There is a nontrivial Euclidean subspace $E \subset \mathbb{R}^{2,n-2}$ such that $\alpha^2_{\varphi|E} = 0$ and α^2_φ is equivalent to the standard Kähler form on the orthogonal complement $E^\perp$ of signature $(2, 2m)$ (again, this is up to conjugation in $SO(2, n-2)$). In this case $Stab_{\alpha^2_\varphi} O(2, n-2) \subset U(1, m) \times O(n - 2(m + 1))$.

Lemma 1.24 implies that the first case occurs iff ker φ is maximal, i.e. 2-dimensional. The second case occurs iff this kernel is one-dimensional whereas the last two cases can only occur if the kernel under Clifford multiplication is trivial. There is no classification of possible higher order algebraic Dirac forms.

Remark 1.26 More properties of α^k_χ in an arbitrary pseudo-Riemannian setting are also discussed in [ACDvP04], where these forms arise in the construction of Lie superalgebra extensions of $\mathfrak{so}(p, q)$.

[9] By this we mean that there are nonzero constants $\mu_i \in \mathbb{R}$ such that $\omega = \sum_{i=1}^m \mu_i e^\flat_{2i-1} \wedge e^\flat_{2i}$. One obtains ω_0, the standard pseudo-Kähler form for $\mu_i = 1$ for all i.

2 Conformal Geometry

Conformal geometry studies smooth manifolds M equipped with an equivalence class $c = [g]$ of pseudo-Riemannian metrics, where we call g and $\widetilde{g}$ conformally equivalent if there is a function $\sigma \in C^\infty(M)$ such that $\widetilde{g} = e^{2\sigma} g$.

2.1 Cartan geometry

Let us briefly introduce our notation for principal bundle theory, following [Bau09]. For M a connected, smooth manifold, a principal G–bundle $\mathcal{P}$ over M will be denoted by $(\mathcal{P}, \pi, M; G)$. When no confusion is likely to occur, we just write $\mathcal{P}$. The right-action of G on $\mathcal{P}$ will be denoted by R. Given a representation $\rho : G \to GL(V)$ of G over a $\mathbb{K}$-vector space V, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the factor space $E := (\mathcal{P} \times V)/G$ under the G-action $(p, v) \cdot g := (p \cdot g, \rho(g^{-1})v)$ together with the natural projection to M turns out to be a $\mathbb{K}$-vector bundle over M, written as

$$E = \mathcal{P} \times_{(G,\rho)} V,$$

and it is called the associated vector bundle. Its elements are classes $[p, v]$, where $p \in \mathcal{P}, v \in V$. Every ρ-invariant bilinear- or sesquilinear form $b_V : V \times V \to \mathbb{K}$ on V induces an inner product b_E on E by

$$b_{E_x}(e, \widetilde{e}) := b_V(v, \widetilde{v}),$$

where $e = [p, v], \widetilde{e} = [p, \widetilde{v}] \in E_x$ for some $p \in P_x$. In particular, every ρ–invariant symmetric ($\mathbb{K} = \mathbb{R}$) resp. Hermitian ($\mathbb{K} = \mathbb{C}$) scalar product $\langle \cdot, \cdot \rangle_V$ on V defines a bundle metric $\langle \cdot, \cdot \rangle_E$ on E and it holds that $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_{E_x}$ have the same signature.
Let $(\mathcal{P}, \pi_{\mathcal{P}}, M; G)$ be a principal G–bundle and let $\lambda : H \to G$ a Lie group homomorphism. Then a λ–reduction of $\mathcal{P}$ is a pair $(\mathcal{Q}, f)$ consisting of a principal H–bundle $(\mathcal{Q}, \pi_{\mathcal{Q}}, M; H)$ and a smooth, equivariant, fibre preserving map $f : \mathcal{Q} \to \mathcal{P}$, i.e.

$$\pi_{\mathcal{P}} \circ f = \pi_{\mathcal{Q}} \text{ and } f(q \cdot h) = f(q) \cdot \lambda(h) \text{ for all } q \in \mathcal{Q} \text{ and } h \in H.$$

In the special case of $H \subset G$ being a Lie subgroup of G and $\lambda = i : H \hookrightarrow G$ being the inclusion, a λ–reduction of $\mathcal{P}$ is also called a H-reduction of $\mathcal{P}$ or a reduction of $\mathcal{P}$ to H.

Given a connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ on the principal bundle $(\mathcal{P}, \pi, M; G)$ and a representation $\rho : G \to GL(V)$ of G over V, a covariant derivative $\nabla^\omega : \Gamma(E) \to \Gamma(T^*M \otimes E)$ on the associated vector bundle $E = \mathcal{P} \times_{(G,\rho)} V$ is induced. It is locally given by

$$(\nabla^\omega_X \sigma)_{|U} = [s, X(v) + \rho_*(\omega(ds(X)))v]. \tag{2.1}$$

Here, $\sigma_{|U} \in \Gamma(E_{|U})$ and $s : U \to \mathcal{P}$ is a local section, $v : U \to V$ is a smooth function such that $\sigma_{|U} = [s, v]$ and $X \in \mathfrak{X}(M)$. If in addition E carries a bundle metric $\langle \cdot, \cdot \rangle_E$ induced by a ρ-invariant scalar product on V , then ∇^ω is metric wrt. $\langle\cdot,\cdot\rangle_E$, i.e. it holds for all sections $\varphi,\psi \in \Gamma(E)$ that

$$X(\langle\varphi,\psi\rangle_E) = \langle\nabla^\omega_X\varphi,\psi\rangle_E + \langle\varphi,\nabla^\omega_X\psi\rangle_E.$$

Moreover, any connection $\omega \in \Omega^1(\mathcal{P},\mathfrak{g})$ gives rise to the holonomy group $Hol_u(\mathcal{P},\omega)$ of ω with respect to u, defined as

$$Hol_u(\mathcal{P},\omega) := \{g \in G \mid \exists \text{ closed path}^1\gamma \text{ in } x \text{ s.th. } P^\omega_\gamma(u) = u\cdot g\},$$

which is a Lie subgroup of G. Here, P^ω_γ denotes the parallel displacement along γ wrt. ω. By requiring that γ is null-homotopic, one obtains the reduced holonomy group $Hol^0_u(\mathcal{P},\omega)$ of ω with respect to u and this turns out to be the identity component of $Hol_u(\mathcal{P},\omega)$. It makes also sense to omit the reference point u and view $Hol(\mathcal{P},\omega)$ as a class of conjugated subgroups of G since M is connected.

Given a vector bundle E over M with covariant derivative ∇, let $\mathcal{P}^\nabla_\gamma$ denote parallel displacement along a path γ. For $x \in M$, the holonomy group of ∇ wrt. x is

$$Hol_x(\nabla) := \{\mathcal{P}^\nabla_\gamma \mid \gamma \text{ closed path in } x \ \} \subset GL(E_x).$$

If $\rho : G \to GL(V)$ is a representation of G over V, $E := \mathcal{P} \times_{(G,\rho)} V$ the associated vector bundle with induced covariant derivative ∇^ω and $x \in M$, then any point $u \in \mathcal{P}_x$ gives rise to a linear fibre isomorphism $[u] : v \in V \to [u,v] \in E_x$. The holonomy groups $Hol_x(E,\nabla^\omega)$ and $Hol_u(\mathcal{P},\omega)$, where $u \in \mathcal{P}_x$, are related by

$$Hol_x(E,\nabla^\omega) = [u] \circ \rho\,(Hol_u(\mathcal{P},\omega)) \circ [u]^{-1}.$$

In particular, if ρ is faithful, these two groups are isomorphic. The holonomy principle interprets parallel tensors on an associated bundle in terms of tensors invariant under the holonomy representation, see [Bau09].

We now turn to Cartan geometries, their connections and holonomy groups, following [CS09]. To this end, let G be a Lie group and $P \subset G$ a closed subgroup. [Sha97] interprets Cartan geometries of type (G,P) as the curved analogues of a homogeneous model, being the homogeneous space G/P together with the canonical $P-$bundle $\pi : G \to G/P$ and its Maurer-Cartan form. This can be made precise in the following way:

Definition 2.1 *Let G be a Lie group and P a closed subgroup. A Cartan geometry of type (G,P) on a smooth manifold M is given by the data $(\mathcal{P},\pi,M,\omega)$, where $\pi : \mathcal{P} \to M$ is a principal fibre bundle with structure group P, endowed with a $\mathfrak{g}$-valued $1-$form $\omega \in \Omega^1(\mathcal{P},\mathfrak{g})$, called the Cartan connection, such that for all $p \in P, u \in \mathcal{P}$ and $X \in \mathfrak{p}$:*

1. $R^*_p(\omega) = Ad(p^{-1}) \circ \omega,$
2. $\omega(\widetilde{X}) = X$, *where* $\widetilde{X} \in \mathfrak{X}(\mathcal{P})$, *given by* $\widetilde{X}(u) = \frac{d}{dt}(u\cdot exp(tX))|_{t=0}$, *is the fundamental vector field generated by* X,
3. $\omega(u) : T_u\mathcal{P} \to \mathfrak{g}$ *is a linear isomorphism.*

[1] In this thesis, a path $\gamma : [a,b] \to M$ is a piecewise smooth curve.

Given a Cartan geometry of type (G,P), there are the so called constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$ for all $X \in \mathfrak{g}$, given by $\omega^{-1}(X)(u) := \omega(u)^{-1}(X)\ \forall u \in \mathcal{P}$. These vector fields lead to a global trivialization of the tangent bundle of $\mathcal{P}$, i.e. $T\mathcal{P} \cong \mathcal{P} \times \mathfrak{g}$, which can be projected to TM by making use of the mapping $pr : \mathcal{P} \times \mathfrak{g} \to TM$, given by $(u,X) \mapsto d\pi_u\big(\omega^{-1}(X)(u)\big)$. As a direct consequence of the definitions, $X \in \mathfrak{p}$ implies that $\omega^{-1}(X) = \widetilde{X}$ and thus, $\omega^{-1}(X)$ is vertical in this case, i.e. it lies in the kernel of $d\pi$. Consequently, pr factors to a map $\mathcal{P} \times \mathfrak{g}/\mathfrak{p} \to TM$, and fixing $u \in \mathcal{P}$ leads to a linear isomorphism $\mathfrak{g}/\mathfrak{p} \to T_{\pi(u)}M$. As ω is P-equivariant, pr factors further to a map $\mathcal{P} \times_P \mathfrak{g}/\mathfrak{p} \to TM$, where P acts on $\mathfrak{g}/\mathfrak{p}$ via restriction of the adjoint action Ad of G on this space[2], and this map is seen to be an isomorphism of vector bundles:

$$TM \cong \mathcal{P} \times_P \mathfrak{g}/\mathfrak{p}$$

Similarly, $T^*M \cong \mathcal{P} \times_P (\mathfrak{g}/\mathfrak{p})^*$, where P acts on $(\mathfrak{g}/\mathfrak{p})^*$ by (restriction of) Ad^*. As a direct consequence, $\dim(M) = \dim(G/P)$, just as for the homogeneous model. Finally, we introduce the curvature of the Cartan connection ω to be

$$\Omega^\omega := d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(\mathcal{P},\mathfrak{g}).$$

Example 2.2 There is always a flat model for Cartan geometries of type (G,P): Let G be a Lie group, P a closed subgroup and let $\pi : G \to G/P$ be the natural projection. We denote by $\omega_G^{MC} \in \Omega^1(G,\mathfrak{g})$ the Maurer Cartan form of G, defined by

$$\omega_G^{MC}(\zeta) := dL_{g^{-1}}(\zeta) \in T_eG \cong \mathfrak{g},$$

where $\zeta \in T_gG$ and $g \in G$. Then $(G,\pi,G/P,\omega_G^{MC})$ is a Cartan geometry of type (G,P), and the Maurer Cartan structure equation precisely states that this Cartan geometry is flat, i.e. $\Omega^{\omega_G^{MC}} = 0$. Furthermore, one can show that the curvature of any Cartan geometry vanishes identically if and only if it is locally isomorphic[3] to a restriction of this homogeneous model ([CS09], Prop. 1.5.2).

Contrary to usual connections, a Cartan connection (1-form) does not allow one to distinguish a connection on $\mathcal{P}$, i.e. a right invariant horizontal distribution on $\mathcal{P}$. Therefore, there is in general no natural notion of induced covariant derivatives on associated vector bundles. However, if the associated vector bundle is formed by restriction of a representation of the larger group G, this is still possible:

Definition 2.3 *Let $\rho : G \to GL(V)$ be a representation of the large group G on a vector space V. Restriction to the subgroup P defines the so called (associated) tractor bundle*

$$\mathcal{W} = \mathcal{P} \times_{(P,\rho)} V.$$

ω induces a covariant derivative on $\mathcal{W}$ in the following way: We can canonically extend the P–principal fibre bundle $\mathcal{P}$ to the G–principal fibre bundle $\overline{\mathcal{P}} := \mathcal{P} \times_P G$ via enlargement, where $\mathcal{P} \times_P G := \{[u,g] \mid (u,g) \in \mathcal{P} \times G\}$, and $[u,g] := \{(u \cdot p, p^{-1}g) \mid p \in P\}$ with G-right action given by $[u,a] \cdot g := [u,ag]$.

[2] In what follows, restrictions of the adjoint action or the action on factor spaces induced by the adjoint action of G will be denoted by the same symbol.

[3] A morphism between two Cartan geometries $(\mathcal{P},\pi,M,\omega)$ and $(\widetilde{\mathcal{P}},\widetilde{\pi},\widetilde{M},\widetilde{\omega})$ of type (G,P) over M is a principal bundle morphism $\phi : \mathcal{P} \to \widetilde{\mathcal{P}}$, satisfying $\phi^*\widetilde{\omega} = \omega$.

Lemma 2.4 ([Feh05], Thm.3.4) *For each Cartan geometry $(\mathcal{P}, \pi, M, \omega)$ of type (G, P) over M, there is a unique principal bundle connection $\overline{\omega} \in \Omega^1(\overline{\mathcal{P}}, \mathfrak{g})$ such that $i^*\overline{\omega} = \omega$, where $i : \mathcal{P} \ni u \mapsto [u, e] \in \overline{\mathcal{P}}$ is the canonical embedding.*

Given an associated tractor bundle defined by a representation $\rho : G \to GL(V)$, we conclude that

$$\mathcal{W} = \mathcal{P} \times_{(P,\rho)} V \cong \overline{\mathcal{P}} \times_{(G,\rho)} V,$$

and thus any Cartan connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ defines a covariant derivative ∇^ω on $\mathcal{W}$ via its extension to the principal bundle connection $\overline{\omega}$ on $\overline{\mathcal{P}}$, namely $\nabla^\omega := \nabla^{\overline{\omega}}$. Using (2.1) one sees that we have the local formula

$$(\nabla^\omega_X \phi)_{|U} = [s, X(v) + \rho_*(\omega(ds(X)))v], \tag{2.2}$$

where $\phi_{|U} = [s, v] \in \Gamma(\mathcal{W}_{|U})$ for a local section $s : U \to \mathcal{P}$[4] and a smooth function $v : U \to V$. Also other parts of the theory developed for induced covariant derivatives on associated vector bundles carry over to this case, and in particular, ∇^ω is metric with respect to bundle metrics on $\mathcal{W}$ induced by G–invariant scalar products on V.

If $(\mathcal{P}, \pi, M, \omega)$ is a Cartan geometry of type (G, P) over M, we define the holonomy of this geometry as the holonomy of the G–principal bundle connection $\overline{\omega}$, which is also very convenient (cf. [Alt12, Feh05, HS11a, HS11b]), i.e.

$$Hol_u(\mathcal{P}, \omega) := Hol_{[u,e]}(\overline{\mathcal{P}}, \overline{\omega}).$$

Remark 2.5 [BJ10] presents a notion of Cartan holonomy, defined via development maps, using the original data $(\mathcal{P}, \omega)$ only and also discusses the precise relation between these two concepts.

2.2 Semi-Riemannian geometry and its conformal behaviour

In this section, let (M, g) be a connected, pseudo-Riemannian manifold of signature (p, q) and dimension $n = p + q \geq 3$. Following [Bau81], TM admits a g–orthogonal decomposition $TM = \tau \oplus \zeta$, where τ is a rank p subbundle and ζ is a rank q subbundle of TM such that the restriction of $-g$ to τ and the restriction of g to ζ are both positive definite. (M, g) is said to be time-orientable, if τ is orientable, and space-orientable, if ζ is orientable. The $GL(n)$-frame bundle $GL(M)$ of M admits an $O(p, q)$–reduction to the $O(p, q)$-bundle of all pseudo-orthonormal frames $\mathcal{P}^g(M)$ (or shortly $\mathcal{P}^g$), given by

$$\mathcal{P}^g(M) := \{(x, (s_1, ..., s_n)) \mid x \in M, (s_1, ..., s_n) \text{ basis of } T_xM, g(s_i, s_j) = \delta_{ij}\epsilon_i\}.$$

In case of orientability this bundle can be further reduced.

Theorem 2.6 ([Bau81]) *Let (M, g) be a pseudo-Riemannian manifold of signature (p, q) with $p > 0$ and pseudo-orthonormal frame bundle $\mathcal{P}^g(M)$. If (M, g) is both space- and time-orientable, then $\mathcal{P}^g$ has four connected components and is reducible to a principal $SO^+(p, q)$-bundle $\mathcal{P}^g_+$ with connected total space, called the bundle of all space- and time-oriented pseudo-orthonormal frames.*

[4] We identify $\mathcal{P}$ with $\mathcal{P} \times_P \{e\} \subset \overline{\mathcal{P}}$ and s with the section $[s, e]$ of $\overline{\mathcal{P}}$.

The Levi-Civita Connection $\nabla^g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ of (M, g) yields the curvature tensor

$$R^g(X,Y) := \nabla^g_X \nabla^g_Y - \nabla^g_Y \nabla^g_X - \nabla^g_{[X,Y]} \text{ for } X, Y \in \mathfrak{X}(M).$$

By means of g, the map R^g can be identified with a $(4,0)$ tensor denoted by the same symbol. Contraction yields the Ricci tensor $Ric^g(X,Y) := tr_g(Z \mapsto R^g(Z,X)Y)$, which can be equivalently viewed as a $(1,1)$ tensor via $g(Ric^g(X),Y) = Ric^g(X,Y)$, and the scalar curvature $R := scal^g := tr_g Ric^g \in C^\infty(M)$. In terms of these data, the Schouten tensor K^g is the $(2,0)$ tensor[5]

$$K^g := \frac{1}{n-2} \cdot \left(\frac{scal^g}{2(n-1)} \cdot g - Ric^g \right),$$

and it can also be considered as an endomorphism $K^g : TM \to T^*M$ by $K^g(X)(Y) := K^g(X,Y)$. The anti-symmetrisation of the covariant derivative of the Schouten tensor defines the (2,1)-Cotton York tensor C^g,

$$C^g(X,Y) := \nabla^g_X(K^g)(Y) - \nabla^g_Y(K^g)(X).$$

In conformal geometry, the Weyl tensor plays an important role and is defined as the trace free part of the Riemannian curvature tensor[6],

$$W^g = R^g - g \owedge K^g.$$

We shall also frequently use the fact that there is an extension of the Levi-Civita connection (also denoted by ∇^g) to smooth tensor fields, uniquely determined by:

1. For $\alpha \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$ it holds that $(\nabla^g_X \alpha)(Y) = \nabla^g_X(\alpha(Y)) - \alpha(\nabla^g_X Y)$.
2. $\nabla^g(\varphi \otimes \psi) = (\nabla^g \varphi) \otimes \psi + \varphi \otimes \nabla^g \psi$ for tensor fields $\varphi \in \Gamma(T^{r_1,s_1}M), \psi \in \Gamma(T^{r_2,s_2}M)$.

For $\alpha \in \Omega^r(M), X, X_1, ..., X_r \in \mathfrak{X}(M)$ we obtain

$$\left(\nabla^g_X \alpha\right)(X_1, ..., X_r) = X(\alpha(X_1, ..., X_r)) + \sum_{i=1}^{r} (-1)^i \alpha(\nabla^g_X X_i, X_1, ..., \widehat{X_i}, ..., X_r),$$

which immediately yields that $\nabla^g_X(Y \lrcorner \alpha) = \left(\nabla^g_X Y\right) \lrcorner \alpha + Y \lrcorner \nabla^g_X \alpha$ for $X, Y \in \mathfrak{X}(M)$
The codifferential d^* is given by $d^*_{|\Omega^k(M)} = (-1)^{n(k-1)+p+1} * d *$, and the operators d and d^* are also expressable in terms of ∇^g and any pseudo-orthonormal basis $(s_1, ..., s_n)$ as

$$d = \sum_{i=1}^{n} \epsilon_i s_i^\flat \wedge \nabla^g_{s_i} \text{ and } d^* = -\sum_{i=1}^{n} \epsilon_i s_i \lrcorner \nabla^g_{s_i}. \tag{2.3}$$

[5]Note that our reference [BJ10] uses a different sign convention for K^g.

[6]Here, $\owedge$ denotes the Kulkarni-Nomizu product: Let b and h be two symmetric $(2,0)-$tensors. We define the $(4,0)-$tensor $b \owedge h$ by

$$(b \owedge h)(X,Y,Z,V) := h(X,Z)b(Y,V) + h(Y,V)b(X,Z) - h(X,V)b(Y,Z) - h(Y,Z)b(X,V).$$

∇^g further defines the holonomy group of (M,g) with respect to $x \in M$ as being

$$Hol_x(M,g) := Hol_x(TM,\nabla^g).$$

As ∇^g is metric, it follows that $Hol_x(M,g)$ is a Lie subgroup of $O(T_xM,g_x)$. We may also consider the tangent bundle as associated bundle $TM \cong \mathcal{P}^g \times_{(O(p,q),\rho)} \mathbb{R}^n$, where $\rho : O(p,q) \to GL(\mathbb{R}^{p,q})$ is the standard representation, and using this, ∇^g corresponds to a connection $\omega^g \in \Omega^1(\mathcal{P}^g, \mathfrak{so}(p,q))$ on $\mathcal{P}^g$. $Hol_x(M,g)$ and $Hol_u(\omega^g)$, where $u \in \mathcal{P}^g_x$, are isomorphic, and if we are only interested in the holonomy group up to conjugation in $O(p,q)$, we will often not distinguish between these groups.

Let us now elaborate on conformal transformations of pseudo-Riemannian structures:

Definition 2.7 *Let M be a smooth manifold of dimension $n = p+q$. Two pseudo-Riemannian metrics g and $\widetilde{g}$ of signature (p,q) on M are called equivalent if $\widetilde{g} = f \cdot g$ for some smooth, positive function $f \in C^\infty(M)$. In this situation, we also say that $\widetilde{g}$ arises from g through a conformal change of the metric. A conformal structure c on M is an equivalence class $[g]$ of metrics. M together with a fixed conformal structure, i.e. the pair (M,c), is called a conformal manifold.*

Remark 2.8 In the following, when referring to conformal structures or conformal manifolds (M,c) , we always assume that $n = p+q \geq 3$. Since multiplication by a positive function does not change the signature of the metric, the signature of $[g]$ is well-defined. Causality is also preserved by conformal equivalence. Whence, it makes sense to speak about space-and time oriented conformal structures (M,c).

Given a conformal change of the metric, that is $\widetilde{g} = e^{2\sigma}g$ for some function $\sigma \in C^\infty(M)$, most curvature tensors show a complicated transformation behaviour (cf. [Fis13]). However, the Weyl tensor (considered as a $(4,0)$ tensor) changes with the same factor, i.e. $W^{\widetilde{g}} = e^{2\sigma}W^g$, and a pseudo-Riemannian manifold (M,g) of dimension ≥ 4 satisfies $W^g = 0$ if and only if (M,g) is locally conformally flat, i.e. for every $x \in M$ there is a chart (U,φ) such that $\varphi : (U,g) \to (\varphi(U), \langle\cdot,\cdot\rangle_{p,q})$ is a conformal diffeomorphism. In dimension 3 one has $W^g = 0$ and C^g is the obstruction against conformal flatness. In dimension 2 every manifold is locally conformally flat.

Given a conformal manifold (M,c), we call a frame $(s_1,...,s_n)$ over $x \in M$ a conformal frame if there is $g \in c$ such that the vectors $s_1,...,s_n$ are pseudo-orthonormal with respect to this metric. Collecting all these frames, we obtain the conformal frame bundle

$$(\mathcal{P}^0, \pi^0, M; CO(p,q)),$$

being a principal bundle with structure group the (linear) conformal group

$$CO(p,q) = \{A \in GL(n,\mathbb{R}) \mid \exists \theta > 0 : \langle Ax, Ay\rangle_{p,q} = \theta\langle x,y\rangle_{p,q} \forall x,y \in \mathbb{R}^n\} \cong \mathbb{R}^+ \times O(p,q).$$

Thus, a conformal structure on M is in terms of G–structures equivalently described as a reduction of the frame bundle $GL(M)$ of M to the linear conformal group $CO(p,q) \subset GL(n,\mathbb{R})$. One could now introduce objects of conformal geometry by defining them with respect to some metric in the conformal class and then show that this definition is independent of the chosen metric. However, the tractor machinery, as to be reviewed in this chapter, shall allow a more elegant approach.

2.3 Semi-Riemannian spin-geometry and its conformal behaviour

In this section, we consider spin structures on pseudo-Riemannian manifolds from a viewpoint of conformal geometry. Main references are [BFGK91, Fri00] (the Riemannian case) and [Bau81] (the general case). Let (M,g) be a be a connected, space- and time-oriented pseudo Riemannian manifold of dimension $n = p+q \geq 3$. In this case, the bundle $\mathcal{P}^g$ of pseudo-orthonormal frames admits a reduction to the bundle $\mathcal{P}^g_+$ of space- and time-oriented pseudo-orthonormal frames with structure group $SO^+(p,q)$.

Definition 2.9 *A spin structure on a space- and time-oriented pseudo-Riemannian manifold (M,g) is a $\lambda-$reduction $(\mathcal{Q}^g_+, f^g)$ of $\mathcal{P}^g_+$, where $\lambda : Spin^+(p,q) \to SO^+(p,q)$ is the double covering map, i.e. a $Spin^+(p,q)$-principal bundle $\mathcal{Q}^g_+$ over M together with a smooth map $f^g : \mathcal{Q}^g_+ \to \mathcal{P}^g_+$ compatible with projections and group actions in the following sense:*

$$\begin{aligned}
&\pi_{\mathcal{P}^g_+} \circ f^g = \pi_{\mathcal{Q}^g_+}, \\
&f^g(u \cdot A) = f^g(u) \cdot \lambda(A) \ \forall u \in \mathcal{Q}^g_+, A \in Spin^+(p,q).
\end{aligned}$$

(M,g) together with a fixed spin structure $(\mathcal{Q}^g_+, f^g)$ is called a spin manifold.

Existence and uniqueness of spin structures on pseudo-Riemannian manifolds is considered in [Bau81]. In this thesis we mainly deal with local results, and locally there is always a uniquely determined spin structure. For the rest of this chapter, let a fixed spin manifold (M,g) with spin structure $(\mathcal{Q}^g_+, f^g)$ be given. $\mathcal{Q}^g_+$ is always connected and f^g is a double covering map in each fibre.

Using the spinor representation, we form the (real or complex) spinor bundle

$$S^g := \mathcal{Q}^g_+ \times_{Spin^+(p,q)} \Delta_{p,q},$$

which is a real (also denoted by $S^g_{\mathbb{R}}$) or complex ($S^g_{\mathbb{C}}$) vector bundle over M, and we call its elements spinors. Its sections, i.e. elements of $\Gamma(S^g)$, are called real resp. complex spinor fields. With the results from the first chapter the spinor bundle splits into $S^g = S^{g,+} \oplus S^{g,-}$ according to the half-spinor representations (if they exist), and therefore, we can talk about half-spinor fields in this case. The spinor algebra as developed before enables us to define the following objects (for proofs cf. [Bau81]):

- The Levi-Civita connection, viewed as a 1-form $\omega^g \in \Omega^1(\mathcal{P}^g_+, \mathfrak{so}(p,q))$, admits a unique lift to a connection $\widetilde{\omega^g} \in \Omega^1(\mathcal{Q}^g_+, \mathfrak{spin}(p,q))$ such that the diagram

$$\begin{tikzcd}
T\mathcal{Q}^g_+ \arrow[r, "\widetilde{\omega^g}"] \arrow[d, "df^g"'] & \mathfrak{spin}(p,q) \arrow[d, "\lambda_*"] \\
T\mathcal{P}^g_+ \arrow[r, "\omega^g"'] & \mathfrak{so}(p,q)
\end{tikzcd}$$

  commutes. Consequently, there is an induced covariant derivative ∇^{S^g} on S^g, the spinor derivative, locally given as follows: Let $s = (s_1, ..., s_n)$ be a local section in

$\mathcal{P}^g_+$. Now let $\widehat{s}$ be a lift of s to $\mathcal{Q}^g_+$. Then each spinor field $\psi \in \Gamma(S^g)$ can locally be written as $\psi = [\widehat{s}, v]$, and it holds that

$$\nabla^{S^g}_X \psi = \left[\widehat{s}, X(v) + \frac{1}{2}\sum_{i<j} \epsilon_i\epsilon_j g(\nabla^g_X s_i, s_j) e_i \cdot e_j \cdot v\right].$$

From this formula we see that the spinor derivative ∇^{S^g} respects the splitting into half-spinor bundles (provided that they exist).

- A nondegenerate inner product $\langle\cdot,\cdot\rangle_{S^g}$ on the spinor bundle is induced by[7]

$$\langle\cdot,\cdot\rangle_{S^g} : S^g \times S^g \to \mathbb{K},\ [\widehat{s}, v] \times [\widehat{s}, w] \mapsto [\widehat{s}, \langle v, w\rangle_{\Delta_{p,q}}].$$

- We note that $TM \cong \mathcal{Q}^g_+ \times_\lambda \mathbb{R}^n$, and this defines Clifford multiplication, in the following also denoted by μ or $\cdot$ or cl, $TM \otimes_{\mathbb{R}} S^g \to S^g$, $[q, x] \otimes [q, v] \mapsto [q, x \cdot v]$. Due to Lemma 1.10, this is well-defined.

The tuple $(S^g, \langle\cdot,\cdot\rangle_{S^g}, \cdot, \nabla^{S^g})$ canonically associated to the spin manifold (M, g) satisfies the following rules and compatibility conditions (cf. [Bau81]):

$$\begin{aligned} (X \cdot Y + Y \cdot X) \cdot \varphi &= -2g(X, Y)\varphi, \\ \langle X \cdot \varphi, \phi\rangle_{S^g} &= (-1)^{p+1}\langle \varphi, X \cdot \phi\rangle_{S^g}, \\ X\langle \varphi, \phi\rangle_{S^g} &= \langle \nabla^{S^g}_X \varphi, \phi\rangle_{S^g} + \langle \varphi, \nabla^{S^g}_X \phi\rangle_{S^g}, \\ \nabla^{S^g}_X (Y \cdot \varphi) &= (\nabla^g_X Y) \cdot \varphi + Y \cdot \nabla^{S^g}_X \varphi, \end{aligned} \tag{2.4}$$

for $X, Y \in \mathfrak{X}(M), \varphi, \phi \in \Gamma(S^g)$. With the previous definition we can also extend Clifford multiplication to forms in the obvious way, namely for $\alpha \in \Omega^k(M)$ and $\varphi \in \Gamma(S^g)$ we set

$$\alpha \cdot \varphi := \sum_{1<i_1<...<i_k\leq n} \epsilon_{i_1}...\epsilon_{i_k} \alpha(s_{i_1}, ..., s_{i_k}) s_{i_1} \cdot ... \cdot s_{i_k} \cdot \varphi,$$

where $(s_1, ..., s_n)$ is some local pseudo-orthonormal frame[8]. As a generalization of (2.4) we then have that (cf. [LM89])

$$\nabla^{S^g}_X (\alpha \cdot \varphi) = (\nabla^g_X \alpha) \cdot \varphi + \alpha \cdot \nabla^{S^g}_X \varphi. \tag{2.5}$$

Remark 2.10 In signatures (p, q) with $q-p \equiv 0, 7 \bmod 8$, the discussion from section 1.3, in particular, Remark 1.15, show us that $\Delta^{\mathbb{C}}_{p,q}$ carries a real structure α which commutes with Clifford multiplication and which therefore induces a real structure $\alpha : S^g_{\mathbb{C}} \to S^g_{\mathbb{C}}$ on the complex spinor bundle $S^g_{\mathbb{C}}$ by setting $\alpha([\widetilde{s}, v]) := [\widetilde{s}, \alpha(v)]$. Consequently, in these signatures we always think of such a real structure fixed and then obtain an inclusion of real spinors $S^g_{\mathbb{R}} \subset S^g_{\mathbb{C}}$ into the space of complex spinors. Let $\varphi = \varphi_{Re} + i \cdot \varphi_{Im} \in \Gamma(S^g)$ be a complex spinor field. Then the local formulas for ∇^{S^g} and Clifford multiplication imply together with the equivariance properties and linearity of α that (cf. [Lei01])

$$(\nabla^{S^g_{\mathbb{C}}}\varphi)_{Re,Im} = \nabla^{S^g_{\mathbb{R}}}\varphi_{Re,Im} \text{ and } (X \cdot \varphi)_{Re,Im} = X \cdot \varphi_{Re,Im}. \tag{2.6}$$

[7] In fact, this is the reason for demanding (M, g) to be time- and space-oriented as otherwise there would be no bundle metric on S^g induced by a scalar product on $\Delta_{p,q}$. Otherwise, one could still define a spin structure in the oriented case to be a $Spin(p, q)$-reduction of $\mathcal{P}^g$.

[8] In the setting of spin manifolds, the term orthonormal frame is always used by us in the sense of time- and space-oriented pseudo-orthonormal.

By construction, Riemannian curvature and spinorial curvature are closely related (cf. [Bau81, LM89]): The endomorphism $R^{S^g}(X,Y) := [\nabla^{S^g}_X, \nabla^{S^g}_Y] - \nabla^{S^g}_{[X,Y]}$ can be expressed as

$$R^{S^g}(X,Y)\varphi = \frac{1}{2}\sum_{i<j}\epsilon_i\epsilon_j R^g(X,Y,s_i,s_j)s_i \cdot s_j \cdot \varphi. \tag{2.7}$$

The composition of ∇^{S^g} with Clifford multiplication defines the Dirac operator

$$D^g : \Gamma(S^g) \overset{\nabla^{S^g}}{\to} \Gamma(T^*M \otimes S^g) \overset{g}{\cong} \Gamma(TM \otimes S^g) \overset{\mu}{\to} \Gamma(S^g).$$

Locally, D^g is given by $D^g = \sum_{i=1}^n \epsilon_i s_i \cdot \nabla^{S^g}_{s_i}$, where $(s_1,...,s_n)$ is some local pseudo-orthonormal frame for (M,g). Performing the spinor covariant derivative ∇^{S^g} followed by orthogonal projection onto the kernel of Clifford multiplication gives rise to a complementary operator acting on spinor fields, the twistor operator P^g

$$P^g : \Gamma(S^g) \overset{\nabla^{S^g}}{\to} \Gamma(T^*M \otimes S^g) \overset{g}{\cong} \Gamma(TM \otimes S^g) \overset{proj_{\ker cl}}{\to} \Gamma(\ker cl).$$

[BFGK91] shows that P^g is locally given by

$$P^g = \sum_{i=1}^n \epsilon_i s_i \otimes \left(\nabla^{S^g}_{s_i} + \frac{1}{n} s_i \cdot D^g\right), \tag{2.8}$$

where s_i is a local pseudo-orthonormal frame. A spinor field $\varphi \in \Gamma(S^g)$ is called a twistor spinor if $\varphi \in \ker P^g$, that is φ satisfies the twistor equation

$$\boxed{\nabla^{S^g}_X \varphi + \frac{1}{n} X \cdot D^g \varphi = 0 \;\; \forall X \in \mathfrak{X}(M)}$$

which is the central object of this thesis. It is easy to deduce that φ satisfies the twistor equation iff there is $\psi \in \Gamma(S^g)$ such that

$$g(X,X)X \cdot \nabla^{S^g}_X \varphi = \psi \text{ for all } X \in \mathfrak{X}(M) \text{ with } g(X,X) = \pm 1. \tag{2.9}$$

A spinor field $\varphi \in \Gamma(S^g)$ is called parallel if $\nabla^{S^g}\varphi = 0$, and it is called harmonic if $D^g\varphi = 0$. Consequently, the twistor equation yields that a spinor field is parallel if and only if it is harmonic and a twistor spinor at the same time. Special solutions of the twistor equation are given by Killing spinors, being solutions of the field equation $\nabla^{S^g}_X \varphi = \lambda X \cdot \varphi$ for all $X \in TM$ and some $\lambda \in \mathbb{C}\backslash\{0\}$. The complex number λ is called the Killing number of φ and turns out to be either real or purely imaginary. Killing spinors are exactly those twistor spinors which satisfy the eigenvalue equation $D^g\varphi = -n\lambda\varphi$ (cf. [BFGK91]) for the Dirac operator.

If n is even ($\mathbb{K} = \mathbb{C}$) resp. $q - p \equiv 0 \mod 4$ ($\mathbb{K} = \mathbb{R}$) and $\varphi = \varphi_+ + \varphi_- \in \Gamma(S^g)$ is a twistor spinor, then $\varphi_\pm \in \Gamma(S^{g,\pm})$ are twistor spinors as well. Moreover, a direct consequence of the formulas (2.6) is that for $q - p \equiv 0,7 \mod 8$ and $\varphi \in \Gamma(S^g_{\mathbb{C}})$ a complex twistor spinor, its real and imaginary part $\varphi_{Re}, \varphi_{Im} \in \Gamma(S^g_{\mathbb{R}})$ (with fixed real structure as in Remark 2.10) are twistor spinors as well.

Proposition 2.11 ([BFGK91]) *Let $\varphi \in \Gamma(S^g)$ be a twistor spinor, $X \in \mathfrak{X}(M)$ a vector field and $\nu \in \Omega^2(M)$ a 2-form. Then the following integrability conditions and relations to Riemannian curvature hold:*

1. $D^{g,2}\varphi = \frac{1}{4}\frac{n}{n-1}R^g\varphi$
2. $\nabla^{S^g}_X D^g\varphi = \frac{n}{2}K^g(X)\cdot\varphi$
3. $W^g(\nu)\cdot\varphi = 0$
4. $W^g(\nu)\cdot D^g\varphi = nC^g(\nu)\cdot\varphi$
5. $(\nabla^g_X W^g)(\nu)\cdot\varphi = X\cdot C^g(\nu)\cdot\varphi + \frac{2}{n}i_X W^g(\nu)\cdot D^g\varphi$

Twistor spinors can be interpreted as parallel objects as follows: We introduce the direct sum bundle $E^g := S^g \oplus S^g$ together with the covariant derivative

$$\nabla^{E^g}_X := \begin{pmatrix} \nabla^{S^g}_X & \frac{1}{n}X\cdot \\ -\frac{n}{2}K^g(X)\cdot & \nabla^{S^g}_X \end{pmatrix}.$$

As a direct consequence of the twistor equation and Proposition 2.11 we obtain:

Proposition 2.12 ([BFGK91, Boh99]) *A twistor spinor $\varphi \in \Gamma(S^g)$ satisfies $\nabla^{E^g}\begin{pmatrix}\varphi \\ D^g\varphi\end{pmatrix} = 0$. Conversely, if $\begin{pmatrix}\varphi \\ \psi\end{pmatrix}$ is ∇^{E^g}-parallel, then φ is a twistor spinor and $\psi = D^g\varphi$.*

The curvature R^{E^g} of ∇^{E^g} satisfies $R^{E^g}(X,Y)\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = \frac{1}{2}\begin{pmatrix} W^g(X,Y)\cdot\varphi \\ W^g(X,Y)\cdot\psi + nC^g(X,Y)\cdot\varphi\end{pmatrix}$, (cf. [BFGK91]) and using this formula, we deduce that the spin manifold (M,g) is conformally flat if and only if $R^{E^g} = 0$, yielding the next statement:

Proposition 2.13 ([BFGK91]) *The dimension of the space ker P^g of twistor spinors is conformally invariant and bounded by dim ker $P^g \leq 2 \cdot$ rang $S^g = 2^{\lfloor \frac{n}{2}\rfloor + 1} =: d_n$. The maximal dimension dim ker P^g can only occur if (M,g) is conformally flat. Conversely, if (M,g) is simply-connected and conformally flat, then dim ker $P^g = d_n$.*

We are interested in the behaviour of twistor spinors under a conformal change $\widetilde{g} = e^{2\sigma}g$ of the metric which means that we have to identify quantities defined in spinor bundles of two conformally equivalent metrics as presented in [Bau81], chapter 3.2.4. The (connected) $SO^+(p,q)$-bundles $\mathcal{P}^g_+$ and $\mathcal{P}^{\widetilde{g}}_+$ are related by the canonical isomorphism

$$\phi_\sigma : \mathcal{P}^g_+ \to \mathcal{P}^{\widetilde{g}}_+,\ (s_1,...,s_n) \mapsto (e^{-\sigma}s_1,...,e^{-\sigma}s_n).$$

It holds that (M,g) is spin iff $(M,\widetilde{g})$ is spin. More precisely, it is shown in [Bau81] that the fixed spin structure $(\mathcal{Q}^g_+, f^g)$ of (M,g) induces via ϕ_σ a distinguished spin structure $(\mathcal{Q}^{\widetilde{g}}_+, f^{\widetilde{g}})$ of $(M,\widetilde{g})$ and an isomorphism $\widetilde{\phi}_\sigma$ of $Spin^+(p,q)$-principle bundles such that the diagram

$$\begin{tikzcd} \mathcal{Q}^g_+ \arrow[r, "\widetilde{\phi}_\sigma"] \arrow[d, "f^g"'] & \mathcal{Q}^{\widetilde{g}}_+ \arrow[d, "f^{\widetilde{g}}"] \\ \mathcal{P}^g_+ \arrow[r, "\phi_\sigma"'] & \mathcal{P}^{\widetilde{g}}_+ \end{tikzcd} \tag{2.10}$$

commutes. Using this property, we obtain natural identifications

$$\begin{array}{llllllll} \widetilde{\ }: S^g & \to & S^{\widetilde{g}}, & \varphi & = & [\widehat{q}, v] & \mapsto & [\widetilde{\phi}_\sigma(\widehat{q}), v] = \widetilde{\varphi}, \\ \widetilde{\ }: TM & \to & TM, & X & = & [q, x] & \mapsto & [\phi_\sigma(q), x] = e^{-\sigma} X, \end{array}$$

where the second map is an isometry wrt. g and $\widetilde{g}$. With these identifications, the covariant derivative on the spinor bundle, the Dirac operator and the twistor operator transform in the following way (cf. [Bau81]):

$$\begin{aligned} \nabla^{\widetilde{S}}_{\widetilde{X}} \widetilde{\varphi} &= e^{-\sigma} \widetilde{\nabla^S_X \varphi} - \frac{1}{2} e^{-2\sigma} \widetilde{(X \cdot grad(e^\sigma) \cdot \varphi + g(X, grad(e^\sigma)) \cdot \varphi)}, \\ \widetilde{D}\widetilde{\varphi} &= e^{-\frac{n+1}{2}\sigma} \widetilde{\left(D(e^{\frac{n-1}{2}\sigma} \varphi) \right)}, \\ \widetilde{P}\widetilde{\varphi} &= e^{-\frac{\sigma}{2}} \widetilde{\left(P(e^{-\frac{\sigma}{2}} \varphi) \right)}. \end{aligned} \tag{2.11}$$

As a consequence, we notice here the conformal covariance of the twistor equation implied by equation (2.11): Obviously, $\varphi \in \Gamma(S^g)$ is a twistor spinor with respect to g if and only if the rescaled spinor $e^{\frac{\sigma}{2}} \widetilde{\varphi} \in \Gamma(S^{\widetilde{g}})$ is a twistor spinor with respect to $\widetilde{g}$. Therefore, the right setting of investigating the twistor equation is not a pseudo-Riemannian spin manifold but rather a conformal spin manifold. In order to make this notion more precise, we introduce the conformal spin group

$$CSpin(p,q) := \mathbb{R}^+ \times Spin(p,q)$$

with the obvious group structure and its identity component $CSpin^+(p,q) = \mathbb{R}^+ \times Spin^+(p,q)$. This group comes together with a double covering map

$$\lambda^0 : CSpin(p,q) \to CO(p,q)\ ,\ \lambda^0 := id \times \lambda. \tag{2.12}$$

Definition 2.14 *Let (M,c) be a space- and time-oriented conformal manifold of signature (p,q) with $n = p+q \geq 3$. A conformal spin structure of (M,c) is a λ^0-reduction of the bundle $\mathcal{P}^0_+$, i.e. a $CSpin^+(p,q)$ principal bundle $(\mathcal{Q}^0_+, \widetilde{\pi}^0, M; CSpin^+(p,q))$ together with a smooth map $f^0 : \mathcal{Q}^0_+ \to \mathcal{P}^0_+$ compatible with projections and group actions in the sense that*

$$\begin{aligned} f^0(q \cdot A) &= f^0(q) \cdot \lambda^0(A)\ \ \forall q \in \mathcal{Q}^0_+, A \in CSpin^+(p,q), \\ \pi^0 \circ f^0 &= \widetilde{\pi}^0. \end{aligned}$$

(M,c) together with a fixed conformal spin structure is called a conformal spin manifold.

Lemma 2.15 ([BJ10]) *(M,c) is a conformal spin manifold if and only if (M,g) is a spin manifold for every metric $g \in c$.*

Proof. Let $(\mathcal{Q}^0_+, f^0)$ be a conformal spin structure of (M,c). By reduction, this induces a metric spin structure $(\mathcal{Q}^g_+, f^g)$ for $g \in c$:

$$\mathcal{Q}^g_+ := (f^0)^{-1}(\mathcal{P}^g_+) \text{ and } f^g := f^0_{|\mathcal{Q}^g_+}.$$

On the other hand, given a spin structure for $g \in c$ and using the fact that $\mathcal{P}^0_+ \cong \mathcal{P}^g_+ \times_{SO^+(p,q)} CO^+(p,q)$, this metric spin structure induces a conformal spin structure of (M,c) by enlargement:

$$\mathcal{Q}^0_+ := \mathcal{Q}^g_+ \times_{Spin^+(p,q)} CSpin^+(p,q) \text{ and } f^0 := f^g \times \lambda.$$

□

Remark 2.16 Let (M,c) is a conformal spin manifold, g and $\widetilde{g} \in c$. If we consider the resulting metric spin structures for g and $\widetilde{g}$ as in the last Proposition, it is clear from the above construction that they are related by the commutative diagram (2.10).

2.4 The first prolongation of a conformal structure

The subsequent construction for conformal structures has been studied intensively in the literature, either in a very explicit approach (cf. [BJ10, Feh05]), or within the general framework of parabolic geometries as in [CS09]. We mainly follow the first method:

The aim of this section is to model conformal structures of signature (p,q) as Cartan geometries of type (B,P) for some group B with Lie algebra $\mathfrak{b}$ and closed subgroup P. We start with investigating the flat model for conformal structures in terms of Cartan geometries. As a reference see [BJ10], section 2.2.1 and references therein or chapter 5 in [Feh05]. In the Riemannian case, the flat models (B,P) for conformal Riemannian structures are the groups of all conformal diffeomorphisms of S^n and the stabilizer of a point. This can be generalized to arbitrary signatures by making use of the following construction: Consider the isotropic cone

$$C^{p,q} := \{x \in \mathbb{R}^{p+1,q+1} \backslash \{0\} \mid \langle x,x\rangle_{p+1,q+1} = 0\}$$

and the projectivization $Q^{p,q} := \mathbb{P}C^{p,q}$, equipped with a conformal structure

$$c := [\mu^* \langle\cdot,\cdot\rangle_{p+1,q+1}]$$

of signature (p,q), where $\mu : Q^{p,q} \to C^{p,q}_+$ is an arbitrary section into a fixed component of the cone. c does not depend on the choice of this section ([Feh05], chapter 5.1). The conformal manifold $(Q^{p,q}, c)$ is called Möbius sphere of signature (p,q), and as elaborated in [Lei01, Fis13], it is conformally diffeomorphic to $(S^p \times S^q)/\mathbb{Z}_2$, equipped with the (projection of) the conformal class $[-g_p + g_q]$, where g_k is the Riemannian standard metric on S^k. This turns out to be a conformal compactification of $\mathbb{R}^{p,q}$. In order to describe $Q^{p,q}$ in terms of Cartan geometries, the Möbius group $B := PO(p+1,q+1) := O(p+1,q+1)/\mathbb{Z}_2$ comes into play and it is isomorphic to the group $Conf(Q^{p,q}, c)$ of all conformal diffeomorphisms of the Möbius sphere and acts transitively and effectively on $Q^{p,q}$. We denote by $Stab_{p_\infty} B \subset B$ the stabilizer of the isotropic line $p_\infty := \mathbb{R}e_- \in Q^{p,q}$ with respect to this action, and one shows that this (parabolic) subgroup can also be realised as subgroup of $O(p+1,q+1)$. It holds that $Q^{p,q} \cong B/P$. Thus, one has:

Proposition 2.17 ([Feh05], Cor. 5.1) *Letting $\pi : B \to Q^{p,q}$ denote the canonical projection, it holds that $(B, \pi, Q^{p,q}; P)$ is a P-principal bundle over $Q^{p,q}$ and $(Q^{p,q}, c)$ is a conformally flat homogeneous space.*

In light of this result, the pair (B, P) can be viewed as the flat model of conformal geometry (cf. Example 2.2). [CS09] shows that every conformal structure over M can be *equivalently* described in terms of a (parabolic) Cartan geometry of type (B, P) over M with B being the Möbius group. Given a conformal structure over a smooth manifold M, we want to present an explicit construction of this associated parabolic geometry of type (B, P) by making use of a process called first prolongation of the conformal frame bundle (cf. [Feh05], chapter 5). This needs some algebraic preparation. Again, we work with the basis $(e_-, e_1, ..., e_n, e_+)$ of $\mathbb{R}^{p+1,q+1}$ introduced in Remark 1.1.
The semisimple Lie algebra $\mathfrak{b} := \mathfrak{o}(p+1, q+1)$ is $|1|$-graded:

$$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \tag{2.13}$$

In terms of matrices, this grading is given by

$$\begin{aligned} \mathfrak{o}(p+1,q+1) &= \left\{ M(y,(a,A),x) := \begin{pmatrix} -a & x & 0 \\ y & A & -x^\sharp \\ 0 & -y^\flat & a \end{pmatrix} \mid \begin{matrix} x \in (\mathbb{R}^n)^*, y \in \mathbb{R}^n \\ a \in \mathbb{R}, A \in \mathfrak{o}(p,q) \end{matrix} \right\}, \\ \mathfrak{b}_{-1} &= \{M(y,(0,0),0) \mid y \in \mathbb{R}^n\} \cong \mathbb{R}^n, \\ \mathfrak{b}_0 &= \{M(0,(a,A),0) \mid a \in \mathbb{R}, A \in \mathfrak{o}(p,q)\} \cong \mathfrak{o}(p,q) \oplus \mathbb{R} \cong \mathfrak{co}(p,q), \\ \mathfrak{b}_1 &= \{M(0,(0,0),x) \mid x \in (\mathbb{R}^n)^*\} \cong (\mathbb{R}^n)^*. \end{aligned} \tag{2.14}$$

In this picture, the commutators $[\cdot,\cdot]$ are given by ([Feh05], Thm. 4.2)

$$\begin{aligned} [\cdot,\cdot] &: \mathfrak{b}_0 \times \mathfrak{b}_0 \to \mathfrak{b}_0, [(A,a),(B,b)] = ([A,B]_{\mathfrak{o}(p,q)}, 0), \\ [\cdot,\cdot] &: \mathfrak{b}_0 \times \mathfrak{b}_{-1} \to \mathfrak{b}_{-1}, [(A,a),y] = Ay + ay, \\ [\cdot,\cdot] &: \mathfrak{b}_1 \times \mathfrak{b}_0 \to \mathfrak{b}_1, [x,(A,a)] = xA + ax, \\ [\cdot,\cdot] &: \mathfrak{b}_{-1} \times \mathfrak{b}_1 \to \mathfrak{b}_0, [y,x] = (yx - J_{p,q}(yx)^T J_{p,q}, xy). \end{aligned}$$

In particular, one has that $[\mathfrak{b}_i, \mathfrak{b}_j]_{\mathfrak{b}} \subset \mathfrak{b}_{i+j}$. In terms of matrices, the stabilizer subgroup $P = Stab_{\mathbb{R}e_-}B$ is isomorphic, under the projection $O(p+1,q+1) \to B$ to the subgroup $Stab_{\mathbb{R}^+e_-}O(p+1,q+1)$ of $O(p+1,q+1)$:

$$P \cong \left\{ Z(a,A,v) := \begin{pmatrix} a^{-1} & v & -\frac{1}{2}a\langle v,v\rangle_{p,q} \\ 0 & A & -aAv^\sharp \\ 0 & 0 & a \end{pmatrix} \;\middle|\; \begin{matrix} a \in \mathbb{R}^+ \\ A \in O(p,q) \\ v \in (\mathbb{R}^n)^* \end{matrix} \right\} \subset O(p+1,q+1). \tag{2.15}$$

This is a parabolic subgroup of B with Lie algebra $\mathfrak{p} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$. It can be further decomposed into a semidirect product $P = B_0 \ltimes_\rho B_1$, with $LA(B_0) = \mathfrak{b}_0$, $LA(B_1) = \mathfrak{b}_1$ and

$$\begin{aligned} B_0 &= \{X(a,A) := Z(a,A,0) \mid a \in \mathbb{R}^+, A \in O(p,q)\} \cong \mathbb{R}^+ \times O(p,q) \cong CO(p,q) \subset P, \\ B_1 &= \{Y(y) := Z(0,0,y) \mid y \in \mathbb{R}^n\} \cong \mathbb{R}^n. \end{aligned}$$

Here, $\rho : B_0 \to Aut(B_1)$ is the conjugation, i.e. $\rho(b_0)(b_1) := b_0 b_1 b_0^{-1}$. In particular, we obtain an embedding

$$i_c : CO(p,q) \hookrightarrow O(p+1,q+1),\ (a,A) \mapsto X(a,A). \tag{2.16}$$

Geometrically, projecting from P to $B_0 \cong P/B_1$ corresponds to the mapping

$$\{\phi \in Conf(Q^{p,q},c) \mid \phi(\mathbb{R}e_-) = \mathbb{R}e_-\} \ni f \mapsto df_{\mathbb{R}e_-} \in CO(T_{\mathbb{R}e_-}Q^{p,q}).$$

Under $B_0 \cong CO(p,q)$ and $\mathfrak{b}_{-1} \cong \mathbb{R}^{p,q}$, the adjoint action of B_0 on $\mathfrak{b}_{-1}$ equals the standard action of $CO(p,q)$ on $\mathbb{R}^{p,q}$: Let $b_0 = X(a,A) \in B_0$ and $v \in \mathfrak{b}_{-1}$, then $Ad(b_0)v = aAv$.

These technical preparations enable us to describe conformal structures as Cartan geometries of type (B,P) as follows: Let (M,c) be a conformal manifold of signature (p,q) with $n = p+q$ and let $\pi^0 : \mathcal{P}^0 \to M$ be the associated $CO(p,q)$–conformal frame bundle. We obtain an identification

$$TM \cong \mathcal{P}^0 \times_{CO(p,q)} \mathbb{R}^n \cong \mathcal{P}^0 \times_{(B_0,Ad)} \mathfrak{b}_{-1},$$

with fibre isomorphisms $[u] : \mathfrak{b}_{-1} \to T_xM,\ v \mapsto [u,v]$, for $u \in \mathcal{P}^0_x$, which give rise to the displacement form $\theta \in \Omega^1(\mathcal{P}^0, \mathfrak{b}_{-1})$, defined by

$$\theta_u(X) := [u]^{-1} d\pi^0_u(X).$$

As $\theta_{|H} : H \to \mathfrak{b}_{-1}$ is an isomorphism for every horizontal space[9] $H \subset T_u\mathcal{P}^0$, we can define the torsion t of $\mathcal{P}^0$,

$$\begin{aligned} t : \{H \mid H \subset T_u\mathcal{P}^0 \text{ horizontal for some } u \in \mathcal{P}^0\} &\to \Lambda^2(\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_{-1}, \\ t(H)(v,w) &:= d\theta_u((\theta_{|H})^{-1}(v), (\theta_{|H})^{-1}(w)) \end{aligned}$$

and call a horizontal space $H \subset T_u\mathcal{P}^0$ torsion free if $t(H) = 0$, The first prolongation of the conformal frame bundle $\mathcal{P}^0$ is then the set

$$\mathcal{P}^1 := \{H \subset T_u\mathcal{P}^0 \mid u \in \mathcal{P}^0, H \text{ horizontal}, t(H) = 0\},$$

which comes together with natural projections

$$\begin{aligned} &\pi^+ : \mathcal{P}^1 \to \mathcal{P}^0, \quad H \mapsto u \text{ for } H \subset T_u\mathcal{P}^0, \\ &\pi^1 : \mathcal{P}^1 \to M, \qquad\quad \pi := \pi^0 \circ \pi^+. \end{aligned}$$

Lemma 2.18 ([Feh05], Thm. 5.6) *$\pi^1 : \mathcal{P}^1 \to M$ is a principal bundle with structure group P and $\pi^+ : \mathcal{P}^1 \to \mathcal{P}^0$ is a principal bundle with structure group B_1. Here, the action of P on $\mathcal{P}^1$ is given by*

$$H \cdot p := \left\{ dR_{b_0}((\theta_{|H})^{-1}(Ad(b_0) \circ \theta(X))) + \widetilde{[Z, \theta(X)]_{\mathfrak{b}}}(u \cdot b_0) \mid X \in H \right\} \subset T_{u \cdot b_0}\mathcal{P}^0, \tag{2.17}$$

where $H \subset T_u\mathcal{P}^0$, $H \in \mathcal{P}^1$ and $p = (b_0, exp(Z)) \in P \cong B_0 \ltimes B_1$.

Remark 2.19 The geometric meaning of (2.17) is that an action by $b_0 \in B_0$ *transports* $H \subset T_u\mathcal{P}^0$ to the point $u \cdot b_0$, whereas $exp(Z) \in B_1$ *rotates* H inside $T_u\mathcal{P}^0$.

[9] By a horizontal subspace $H \subset T_u\mathcal{P}^0$ we mean a subspace that is complementary to the vertical tangent space $\ker d\pi^0_u \subset T_u\mathcal{P}^0$.

2.5 The normal conformal Cartan connection

It is possible to distinguish a Cartan connection on the principal bundle $\mathcal{P}^1$ in order to obtain a Cartan geometry which equivalently describes our underlying conformal structure. Let $\omega \in \Omega^1(\mathcal{P}^1, \mathfrak{b})$ be any Cartan connection. It splits with respect to the grading of $\mathfrak{b}$,

$$\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1.$$

Definition 2.20 *A Cartan connection $\omega \in \Omega^1(\mathcal{P}^1, \mathfrak{b})$ is called admissible if its components satisfy the following conditions:*

1. $\omega_{-1} = (\pi^+)^*\theta$, *where θ is the displacement form on $\mathcal{P}^0$.*
2. $d\pi^+_H(\zeta) - (\omega_0)\widetilde{{}_{H}(\zeta)}(u) \in H \subset T_u\mathcal{P}^0 = H \oplus d\pi^0_u\mathcal{P}^0$ *for all $\zeta \in T_H\mathcal{P}^1$.*

Imposing an additional trace condition on the curvature $\Omega^\omega \in \Omega^2(\mathcal{P}^1, \mathfrak{b})$ distinguishes a unique admissible Cartan connection. To this end, consider the codifferential operator

$$\partial^*_\omega : \Omega^2(\mathcal{P}^1, \mathfrak{b}) \to \Omega^1(\mathcal{P}^1, \mathfrak{b}),$$

given by $(\partial^*_\omega \eta)_H(X) := \sum_{i=1}^n \left[v_i^*, \eta\left(X, \omega_H^{-1}(v_i)\right)\right]_{\mathfrak{g}}$, where $H \in \mathcal{P}^1$, $X \in T_H\mathcal{P}^1$, $(v_1, ..., v_n)$ is a basis in $\mathfrak{b}_{-1} \cong \mathbb{R}^n$ and $(v_1^*, ..., v_n^*)$ denotes the dual basis in $\mathfrak{b}_1 \cong (\mathbb{R}^n)^*$.

Proposition 2.21 ([Feh05], chapter 6) *There is a unique admissible Cartan connection $\omega^{nc} \in \Omega^1(\mathcal{P}^1, \mathfrak{b})$ such that $\partial^*_{\omega^{nc}}\Omega^{\omega^{nc}} = 0$. ω^{nc} is called the normal conformal Cartan connection (nc-Cartan connection).*

Remark 2.22 In the language of [CS09] we started with a conformal structure c on M and constructed a normal, regular parabolic geometry $(\mathcal{P}^1, \pi, M, \omega^{nc})$ of type (B, P) over M which, as shown in [CS09], equivalently describes this structure in the following sense: Given a normal parabolic geometry $(\mathcal{P}, \pi, M, \omega)$ of type (B, P), we can use the theory of infinitesimal flag structures as developed in [CS09] to see that there is a conformal structure on M associated to this parabolic geometry with $\mathcal{P}^0 = \mathcal{P}/B^1$, and by construction, these two directions are inverse to each other (up to isomorphism in the category of parabolic geometries), for details cf. [CS09].

Remark 2.23 In what follows, we view ω^{nc} as a Cartan connection of type $(O(p+1, q+1), P)$. This can be done because of the following: By definition, a Cartan connection of type (B, P) depends on the data P and $\mathfrak{b}$ only. Our chosen parabolic subgroup P in $O(p+1, q+1)/\mathbb{Z}_2$ can also be realised as a closed subgroup P of $O(p+1, q+1)$ as seen before. Explicitly, P is given as the stabilizer of the isotropic ray $\mathbb{R}^+e_-$ in $O(p+1, q+1)$. Consequently, we may also consider conformal structures as parabolic geometries of type $(O(p+1, q+1), P)$, and we therefore redefine $B = O(p+1, q+1)$. The resulting new flat model $\widehat{Q}^{p,q} \cong B/P$ is then a double cover of $Q^{p,q}$ and it can be viewed as the set of null rays in $\mathbb{R}^{p+1,q+1}$ equipped with a natural and flat conformal structure $\widehat{c}$. For concrete calculations we use a realisation of this flat model presented in [Lei01]: $\widehat{Q}^{p,q}$ is naturally embedded in $\mathbb{R}^{p+1,q+1}$, where $n = p + q$, via

$$i : \widehat{Q}^{p,q} \hookrightarrow \mathbb{R}^{p+1,q+1}, \text{ where } \mathbb{R}_+ \cdot x \mapsto \sqrt{\frac{2}{\langle x, x\rangle_{n+2}}} \cdot x \tag{2.18}$$

and where $\langle\cdot,\cdot\rangle_{n+2}$ denotes the standard Euclidean inner product. One checks that $i(\widehat{Q}^{p,q}) = S^p \times S^q \subset \mathbb{R}^{0,p+1} \times \mathbb{R}^{0,q+1}$. It holds that $\widehat{c} = [i^*\langle\cdot,\cdot\rangle_{p+1,q+1}]$. Thus, the conformally flat manifold $\widehat{C}^{p,q} = (\widehat{Q}^{p,q}, \widehat{c})$ can be identified with $(S^p \times S^q, [-g_p + g_q])$. Note that $\widehat{Q}^{p,q}$ is not connected for $p = 0, n$ and not simply-connected for $p = 1, n-1$.

Remark 2.24 Following [BJ10], section 2.2.3, fixing $g \in c$ defines a $B_0 \cong CO(p,q)$ equivariant smooth section $\eta^g : \mathcal{P}^0 \to \mathcal{P}^1$, $u \mapsto \ker \omega^g_u$ in the B_1-bundle $(\mathcal{P}^1, \pi^+, \mathcal{P}^0; B_1)$ and an $O(p,q)$ equivariant smooth section

$$\sigma^g : \mathcal{P}^g \to \mathcal{P}^1,\ u \mapsto \eta^g(\iota^g(u))$$

in the P–bundle $(\mathcal{P}^1, \pi, M; P)$. Here, $\omega^g \in \Omega^1(\mathcal{P}^g, \mathfrak{o}(p,q))$ is the Levi-Civita connection corresponding to ∇^g (more precisely, we consider its extension to a torsion free connection on $\mathcal{P}^0$) and $\iota^g : \mathcal{P}^g \hookrightarrow \mathcal{P}^0$ is the canonical inclusion. Via η^g and ι^g, every metric $g \in c$ defines a $i_c : CO(p,q) \hookrightarrow O(p+1,q+1)$-reduction of $\mathcal{P}^1$ and $\mathcal{P}^0$ to the bundle $\mathcal{P}^g$. Furthermore, one has the following transition formula, which can be found in the proof of Proposition 2.2.5 in [BJ10]: Let $\widetilde{g} = e^{2\sigma} g$ be a rescaled metric and let $u = (s_1, ..., s_n)$ and $\widetilde{u} = (e^{-\sigma} s_1, ..., e^{-\sigma} s_n)$ be local pseudo-orthonormal bases for g resp. $\widetilde{g}$ over $U \subset M$. Then the transformation formula for the Levi-Civita connection and the definition of the P–action on $\mathcal{P}^1$ show that

$$\sigma^g(u) = \sigma^{\widetilde{g}}(\widetilde{u}) \cdot \begin{pmatrix} e^{-\sigma} & -e^{\sigma}[u]^{-1} d\sigma & -\frac{1}{2} e^{\sigma} \|d\sigma\|^2_g \\ 0 & I_n & [u]^{-1} \mathrm{grad}^g \sigma \\ 0 & 0 & e^{\sigma} \end{pmatrix}. \tag{2.19}$$

Finally, let $x \in M$, $u = (s_1, ..., s_n) \in \mathcal{P}^g_x$ and $X \in T_u \mathcal{P}^g$. Then the local form of ω^{nc} with respect to the fixed metric g is given by

$$((\sigma^g)^* \omega^{nc})_u(X) = \theta_u(X) + \omega^g_u(X) + \sum_{i=1}^n \epsilon_i K^g_x(d\pi^0_u(X), s_i) s_i^\flat. \tag{2.20}$$

Remark 2.25 Given the grading (2.13) of $\mathfrak{o}(p+1, q+1)$, note that there are other possible choices of associated Lie groups. One of them corresponds to oriented conformal structures, namely in the setting of the previous section we could as well choose

$$\begin{aligned} B^+ &:= SO^+(p+1, q+1), \\ P^+ &:= P \cap B^+, \\ B_0^+ &:= B_0 \cap B^+ \cong CO^+(p,q), \\ B_1^+ &:= B_1 \cap B^+, \end{aligned} \tag{2.21}$$

where $CO^+(p,q) = \mathbb{R}^+ \times SO^+(p,q)$ is the identity component of $CO(p,q)$. We will in general assume (M,c) to be space- and time-oriented. In this case, the conformal frame bundle $\mathcal{P}^0$ admits a reduction to the bundle $(\mathcal{P}^0_+, \pi^0, M; CO^+(p,q))$ of time- and space-oriented conformal frames with structure group $CO^+(p,q)$, and for every metric $g \in c$ the bundle $\mathcal{P}^g$ admits a reduction to $\mathcal{P}^g_+$. With the choice (2.21), the theory of first prolongation works completely analogous, where we replace $\mathcal{P}^g$ and $\mathcal{P}^0$ by $\mathcal{P}^g_+$ and $\mathcal{P}^0_+$ respectively, and the the first prolongation $\mathcal{P}^1$ is replaced by the P^+-bundle

$$\mathcal{P}^1_+ := \{H \mid u \in \mathcal{P}^0_+, H \subset T_u \mathcal{P}^0_+ \text{ horizontal }, t(H) = 0\}.$$

Similarly to the non-oriented case, we will view the normal conformal Cartan connection as a Cartan connection of type $(SO^+(p+1, q+1), P^+)$.

2.6 The conformal standard tractor-and tractor form bundle

Given a conformal structure c on M with first prolongation $\mathcal{P}^1$ and a representation $\rho : O(p+1, q+1) \to GL(V)$, we can form the associated tractor bundle $E = \mathcal{P}^1 \times_{(P,\rho)} V$ over the conformal manifold (M, c), and ω^{nc} induces a covariant derivative ∇^{nc} on E in the usual way. We will mainly apply this to ρ being the natural action of $O(p+1, q+1)$ on $\mathbb{R}^{p+1,q+1}$ or $\Lambda^{k+1}_{p+1,q+1}$, yielding the standard tractor bundle,

$$\mathcal{T}(M) := \mathcal{P}^1 \times_P \mathbb{R}^{p+1,q+1},$$

and the tractor (k+1)-form bundle,

$$\Lambda^{k+1}_{\mathcal{T}} M := \mathcal{P}^1 \times_P \Lambda^{k+1}_{p+1,q+1},$$

respectively, where $\Lambda^1_{\mathcal{T}} M = \mathcal{T}^* M$. In the maximally oriented case we will always further reduce to $\mathcal{P}^1_+$. The standard scalar product $\langle \cdot, \cdot \rangle_{p+1,q+1}$ on $\mathbb{R}^{p+1,q+1}$ induces bundle metrics $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ on $\mathcal{T}(M)$ and $\Lambda^{k+1}_{\mathcal{T}}$, and it turns out that ∇^{nc} on $\mathcal{T}(M)$ is metric wrt. $\langle \cdot, \cdot \rangle_{\mathcal{T}}$, i.e.

$$X(\langle \phi, \psi \rangle_{\mathcal{T}}) = \langle \nabla^{nc}_X \phi, \psi \rangle_{\mathcal{T}} + \langle \phi, \nabla^{nc}_X \psi \rangle_{\mathcal{T}},$$

for all sections $\phi, \psi \in \Gamma(\mathcal{T}(M))$ and $X \in \mathfrak{X}(M)$. In this sense, ∇^{nc} can be viewed as conformal analogue of the Levi-Civita connection, making it reasonable to define the conformal holonomy $Hol(M, c)$ of (M, c) in terms of this connection, i.e. for $x \in M$ we set

$$Hol_x(M, c) := Hol_x(\mathcal{T}(M), \nabla^{nc}) \subset O(\mathcal{T}_x(M), \langle \cdot, \cdot \rangle_{\mathcal{T}}) \cong O(p+1, q+1).$$

For a general associated tractor bundle $E = \mathcal{P}^1 \times_{(P,\rho)} V$, fixing a metric $g \in c$ leads to a reduction of $\mathcal{P}^1$ to $\mathcal{P}^g$ and an isomorphism $E \cong \mathcal{P}^g \times_{(O(p,q),\rho)} V$. We want to find the description of ∇^{nc} under this isomorphism. To this end, let $\mathcal{W}$ be any of the bundles $TM \cong \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{b}_{-1}$, $T^*M \cong \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{b}_1$ or $\mathfrak{so}(TM, g) \cong \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{so}(p, q)$. We define an action $\rho^g : \mathcal{W} \to End(E)$ by

$$\rho^g(\Theta)t := \left[u, \rho_* \left([u]^{-1}\Theta\right) [u]^{-1} t\right] \in E_x, \tag{2.22}$$

where $u \in \mathcal{P}^g_x$, $t \in E_x$ and $\Theta \in \mathcal{W}_x$. This does not depend on the choice of the frame u.

Lemma 2.26 ([BJ10], Prop. 2.2.4) *The covariant tractor derivative $\nabla^{nc} : \Gamma(E) \to \Gamma(T^*M \otimes E)$ on $E = \mathcal{P}^1 \times_{(P,\rho)} V$ and its curvature endomorphism $R^{nc}(X,Y) := [\nabla^{nc}_X, \nabla^{nc}_Y] - \nabla^{nc}_{[X,Y]}$ are wrt. a metric $g \in c$ and the resulting isomorphism $E \cong \mathcal{P}^g \times_{(O(p,q),\rho)} V$ given by*

$$\begin{aligned} \nabla^{nc}_X &= \nabla^g_X + \rho^g(X) + \rho^g\left(K^g(X)\right), \\ R^{nc}(X,Y) &= \rho^g(W^g(X,Y)) - \rho^g(C^g(X,Y)). \end{aligned}$$

Let us apply this again to $\mathcal{T}(M)$, i.e. $\rho : O(p+1, q+1) \to GL\left(\mathbb{R}^{p+1,q+1}\right)$ being the standard representation. Restricting ρ to $O(p,q)$ gives rise to the splitting

$$\begin{aligned} \mathbb{R}^{p+1,q+1} &\cong \mathbb{R} \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}, \\ ae_- + Y + be_+ &\mapsto (a, Y, b) \end{aligned}$$

into three $O(p,q)$-representation spaces, where $O(p,q)$ acts trivially on the two 1-dimensional subspaces and by matrix action on $\mathbb{R}^{p,q}$. Thus, fixing $g \in c$ leads to a vector bundle isomorphism

$$\begin{aligned}\Phi^g : \mathcal{T}(M) &\cong \mathcal{I}_- \oplus TM \oplus \mathcal{I}_+ =: \mathcal{T}(M)_g, \\ [\sigma^g(u), ae_- + y + be_+] &\mapsto (a, [u,y], b),\end{aligned} \tag{2.23}$$

where $\mathcal{I}_\pm \cong \underline{M}$ are trivial line bundles over M which admit globally totally lightlike sections $s_\pm$. In particular, we can use Φ^g to identify sections of $\mathcal{T}(M)$ with triples (α, Y, β), where $\alpha, \beta \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$. Under this identification, the bundle metric $\langle(\alpha_1, Y_1, \beta_1), (\alpha_2, Y_2, \beta_2)\rangle_{\mathcal{T}}$ is given by

$$\langle(\alpha_1, Y_1, \beta_1), (\alpha_2, Y_2, \beta_2)\rangle_{\mathcal{T}} = \alpha_1\beta_2 + \alpha_2\beta_1 + g(Y_1, Y_2). \tag{2.24}$$

We call Φ^g the g-trivialization of $\mathcal{T}(M)$ which allows metric representations of tractors.

Lemma 2.27 ([BJ10], Prop. 2.2.6) *Let $g \in c$. Under the identification (2.23), the tractor derivative ∇^{nc} on $\mathcal{T}(M)$, i.e. the map $\Phi^g \circ \nabla^{nc} \circ (\Phi^g)^{-1}$, is given by*

$$\nabla^{nc}_X \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} = \begin{pmatrix} X(\alpha) + K^g(X,Y) \\ \nabla^g_X Y + \alpha X - \beta K^g(X)^\sharp \\ X(\beta) - g(X,Y) \end{pmatrix}, \tag{2.25}$$

and the curvature R^{nc} of ∇^{nc} satisfies

$$R^{nc}(X_1, X_2) \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} = \begin{pmatrix} -C^g(X_1, X_2)Y \\ W^g(X_1, X_2)Y + \beta C^g(X_1, X_2)^\sharp \\ 0 \end{pmatrix}. \tag{2.26}$$

For different metrics $g \in c$ the resulting g-trivializations can be identified as follows:

Lemma 2.28 ([BJ10], Prop. 2.2.5) *Under a conformal change $\widetilde{g} = e^{2\sigma} g$, transformation of the metric representation of a standard tractor, i.e. the map $T(g,\sigma) := \Phi^{\widetilde{g}} \circ (\Phi^g)^{-1}$: $\mathcal{T}(M)_g \to \mathcal{T}(M)_{\widetilde{g}}$ is given by*

$$\begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{Y} \\ \widetilde{\beta} \end{pmatrix} = \begin{pmatrix} e^{-\sigma}(\alpha - Y(\sigma) - \frac{1}{2}\beta \cdot \|grad^g \sigma\|^2_g \\ e^{-\sigma}(Y + \beta \cdot grad^g \sigma) \\ e^{\sigma}\beta \end{pmatrix}. \tag{2.27}$$

We now turn to the tractor $(k+1)$-form bundle $\Lambda^{k+1}_{\mathcal{T}}(M)$ and seek for a similar description of $\nabla^{nc} : \Gamma(\Lambda^{k+1}_{\mathcal{T}}(M)) \to \Gamma(T^*M \otimes \Lambda^{k+1}_{\mathcal{T}}(M))$ wrt. a metric. As it turns out, having fixed a metric $g \in c$, allows us to describe tractor forms in terms of usual differential forms with the help of the following algebraic construction, using the decomposition $\mathbb{R}^{p+1,q+1} \cong \mathbb{R}e_- \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}e_+$. Clearly, every form $\alpha \in \Lambda^{k+1}_{p+1,q+1}$ decomposes into

$$\alpha = e^\flat_+ \wedge \alpha_+ + \alpha_0 + e^\flat_- \wedge e^\flat_+ \wedge \alpha_\mp + e^\flat_- \wedge \alpha_- \tag{2.28}$$

for uniquely determined forms $\alpha_-, \alpha_+ \in \Lambda^k_{p,q}, \alpha_0 \in \Lambda^{k+1}_{p,q}$ and $\alpha_\mp \in \Lambda^{k-1}_{p,q}$. Using this decomposition, the restriction of $\rho : O(p+1,q+1) \to GL\left(\Lambda^{k+1}_{p+1,q+1}\right)$ to $O(p,q) \hookrightarrow O(p+1,q+1)$ defines an isomorphism of $O(p,q)$-modules,

$$\Lambda^{k+1}_{p+1,q+1} \cong \Lambda^k_{p,q} \oplus \Lambda^{k+1}_{p,q} \oplus \Lambda^{k-1}_{p,q} \oplus \Lambda^k_{p,q}.$$

Consequently, fixing a metric $g \in c$ leads to a vector bundle isomorphism

$$\Lambda^{k+1}_{\mathcal{T}}(M) \cong \mathcal{P}^g \times_{O_{p,q}} \left(\Lambda^k_{p,q} \oplus \Lambda^{k+1}_{p,q} \oplus \Lambda^{k-1}_{p,q} \oplus \Lambda^k_{p,q}\right) \cong \left(\mathcal{P}^g \times_{O_{p,q}} \Lambda^k_{p,q}\right) \oplus ... \oplus \left(\mathcal{P}^g \times_{O_{p,q}} \Lambda^k_{p,q}\right),$$

and thus to a natural g-metric representation of the tractor $(k+1)$-form bundle:

$$\Phi^g_\Lambda : \Lambda^{k+1}_{\mathcal{T}}(M) \overset{g}{\to} \Lambda^k(M) \oplus \Lambda^{k+1}(M) \oplus \Lambda^{k-1}(M) \oplus \Lambda^k(M) =: \Lambda^{k+1}_{\mathcal{T}}(M)_g.$$

Applying this pointwise yields that each tractor $(k+1)$-form $\alpha \in \Omega^{k+1}_{\mathcal{T}}(M) := \Gamma\left(\Lambda^{k+1}_{\mathcal{T}}(M)\right)$ uniquely corresponds via $g \in c$ to a set of differential forms,

$$\Phi^g_\Lambda(\alpha) = (\alpha_+, \alpha_0, \alpha_\mp, \alpha_-) \in \Omega^k(M) \oplus \Omega^{k+1}(M) \oplus \Omega^{k-1}(M) \oplus \Omega^k(M). \tag{2.29}$$

We further introduce the g-dependent projections

$$\begin{aligned} proj^g_{\Lambda,+} : \Omega^{k+1}_{\mathcal{T}}(M) &\to \Omega^k(M) \\ \alpha &\mapsto \alpha_+, \text{ where } \Phi^g_\Lambda(\alpha) = (\alpha_+, \alpha_0, \alpha_\mp, \alpha_-) \end{aligned}$$

An equivalent way of expressing the g–metric representation is pointwise application of the isomorphism (2.28) such that

$$\alpha = s_-^\flat \wedge \alpha_- + \alpha_0 + s_-^\flat \wedge s_+^\flat \wedge \alpha_\mp + s_+^\flat \wedge \alpha_+, \tag{2.30}$$

where $s_\pm$ are the lightlike sections in $\mathcal{I}_\pm$ induced via g and $e_\pm$.

Lemma 2.29 *Let (M,c) be a conformal manifold with tractor $(k+1)$-form bundle $\Lambda^{k+1}_{\mathcal{T}}(M)$ and $g \in c$. The operator $\Phi^g_\Lambda \circ \nabla^{nc} \circ \left(\Phi^g_\Lambda\right)^{-1}$, i.e. the description of $\nabla^{nc} : \Gamma(\Lambda^{k+1}_{\mathcal{T}}(M)) \to \Gamma(T^*M \otimes \Lambda^{k+1}_{\mathcal{T}}(M))$ with respect to the identification (2.29) is given by the following matrix expression:*

$$\nabla^{nc}_X \alpha \overset{g}{=} \begin{pmatrix} \nabla^g_X & -X\lrcorner & -X^\flat\wedge & 0 \\ -K^g(X)\wedge & \nabla^g_X & 0 & X^\flat\wedge \\ -\left(K^g(X)\right)^\sharp\lrcorner & 0 & \nabla^g_X & -X\lrcorner \\ 0 & \left(K^g(X)\right)^\sharp\lrcorner & -K^g(X)\wedge & \nabla^g_X \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_0 \\ \alpha_\mp \\ \alpha_- \end{pmatrix}$$

Proof. This is an direct application of Lemma 2.26. Let $\alpha \in \Omega^{k+1}_{\mathcal{T}}(M)$ and $y \in M$. We may write

$$\alpha(y) = \left[\sigma^g(u), e_+^\flat \wedge \widehat{\alpha}_+ + \widehat{\alpha}_0 + e_-^\flat \wedge e_+^\flat \wedge \widehat{\alpha}_\mp + e_-^\flat \wedge \widehat{\alpha}_-\right]$$

for some $u \in \mathcal{P}^g_y$ and a form $\widehat{\alpha} \in \Lambda^{k+1}_{p+1,q+1}$ decomposed as in (2.28). By definition, we have for $\kappa \in \{+,0,\mp,-\}$ that $\alpha_\kappa(y) = [u, \widehat{\alpha}_\kappa]$. Next, we note that the natural action $\circ$ of $\mathfrak{so}(p+1,q+1) \cong \Lambda^2_{p+1,q+1}$ on $\in \Lambda^{k+1}_{p+1,q+1}$ yields the formulas (cf. [Lei05])

$$\begin{aligned} (e_+^\flat \wedge x^\flat) \circ (e_+^\flat \wedge \widehat{\alpha}_+) &= 0, \\ (e_+^\flat \wedge x^\flat) \circ \widehat{\alpha}_0 &= -e_+^\flat \wedge (x\lrcorner\widehat{\alpha}_0), \\ (e_+^\flat \wedge x^\flat) \circ (e_-^\flat \wedge e_+^\flat \wedge \widehat{\alpha}_\mp) &= -e_+^\flat \wedge x^\flat \wedge \widehat{\alpha}_\mp, \\ (e_+^\flat \wedge x^\flat) \circ (e_-^\flat \wedge \widehat{\alpha}_-) &= x^\flat \wedge \widehat{\alpha}_- - e_-^\flat \wedge e_+^\flat \wedge (x\lrcorner\widehat{\alpha}_+), \end{aligned}$$

for $e_+^\flat \wedge x^\flat \in \mathfrak{b}_{-1}$ with corresponding skew-symmetric endomorphism (cf. (1.2)) $\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^\flat & 0 \end{pmatrix}$ which (2.14) identifies with $x \in \mathbb{R}^{p,q}$, and similarly

$$\begin{aligned}
(e_-^\flat \wedge z) \circ (e_+^\flat \wedge \widehat{\alpha}_+) &= z \wedge \widehat{\alpha}_+ + e_-^\flat \wedge e_+^\flat \wedge (z^\sharp \lrcorner \widehat{\alpha}_+), \\
(e_-^\flat \wedge z) \circ \widehat{\alpha}_0 &= -e_-^\flat \wedge (z^\sharp \lrcorner \widehat{\alpha}_0) \\
(e_-^\flat \wedge z) \circ (e_-^\flat \wedge e_+^\flat \wedge \widehat{\alpha}_\mp) &= e_-^\flat \wedge z \wedge \widehat{\alpha}_\mp, \\
(e_-^\flat \wedge z) \circ (e_-^\flat \wedge \widehat{\alpha}_-) &= 0,
\end{aligned}$$

for $e_-^\flat \wedge z \in \mathfrak{b}_1$ with corresponding skew-symmetric endomorphism $\begin{pmatrix} 0 & -z & 0 \\ 0 & 0 & z^\sharp \\ 0 & 0 & 0 \end{pmatrix}$ which (2.14) identifies with $-z \in (\mathbb{R}^{p,q})^*$. Now let $X = [u, x] \in TM \cong \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{b}_{-1}$ and $x \in \mathfrak{b}_{-1} \cong \mathbb{R}^n$. It follows that $K^g(X) = [u, z] \in T^*M \cong \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{b}_1$ for some $z \in \mathfrak{b}_1 \cong (\mathbb{R}^n)^*$. With these preparations, we now have by (2.22) that

$$\rho^g(X)\alpha(y) = \left[u, x \wedge \widehat{\alpha}_- - e_-^\flat \wedge e_+^\flat \wedge (x \lrcorner \widehat{\alpha}_+) - e_+^\flat \wedge (x \lrcorner \widehat{\alpha}_0) - e_+^\flat \wedge x^\flat \wedge \widehat{\alpha}_\mp\right] \overset{\Phi_\Lambda^g}{=} \begin{pmatrix} -X \lrcorner \alpha_0 - X^\flat \wedge \alpha_\mp \\ X \wedge \alpha_- \\ -X \lrcorner \alpha_- \\ 0 \end{pmatrix}(y),$$

$$\begin{aligned}
\rho^g(K^g(X))\alpha(y) &= \left[e_-^\flat \wedge (z^\sharp \lrcorner \widehat{\alpha}_0) - e_-^\flat \wedge z \wedge \widehat{\alpha}_\mp - z \wedge \alpha_+ - e_-^\flat \wedge e_+^\flat \wedge (x \lrcorner \widehat{\alpha}_+)\right] \\
&\overset{\Phi_\Lambda^g}{=} \begin{pmatrix} 0 \\ -K^g(X) \wedge \alpha_+ \\ -(K^g(X))^\sharp \lrcorner \alpha_+ \\ K^g(X)^\sharp \lrcorner \alpha_0 - K^g(X) \wedge \alpha_\mp \end{pmatrix}(y).
\end{aligned}$$

From these formulas, the Proposition follows immediately with Lemma 2.26. □

Finally, we give the behaviour of $proj^g_{\Lambda,+}$ under a conformal change:

Lemma 2.30 *Let $\alpha \in \Omega_{\mathcal{T}}^{k+1}(M)$ be a tractor $(k+1)$–form on (M, c). Fix $g \in c$ and $\widetilde{g} = e^{2\sigma} g \in c$ and let $\alpha_+ = proj^g_{\Lambda,+}\alpha$, $\widetilde{\alpha}_+ = proj^{\widetilde{g}}_{\Lambda,+}\alpha \in \Omega^k(M)$. These forms are related by*

$$\widetilde{\alpha}_+ = e^{(k+1)\sigma} \alpha_+.$$

Proof. Using the standard tractor metric, one dualizes the map (2.27) $T(g, \sigma) : \mathcal{T}(M)_g \to \mathcal{T}(M)_{\widetilde{g}}$ from Lemma 2.28 to $T^*(g, \sigma) : \mathcal{T}^*(M)_g \to \mathcal{T}^*(M)_{\widetilde{g}}$, and extends this naturally to $T^{\Lambda,k+1}(g, \sigma) : \Lambda_{\mathcal{T}}^{k+1}(M)_g \to \Lambda_{\mathcal{T}}^{k+1}(M)_{\widetilde{g}}$. The construction and (2.27) yield that

$$\widetilde{\alpha}_+ = proj^{\widetilde{g}}_{\Lambda,+}\left(T^{\Lambda,k+1}(g, \sigma)(s_+^\flat \wedge \alpha_+)\right).$$

The claim follows by straightforwardly unwinding these definitions. □

2.7 First prolongation of a conformal spin structure

We now assume that (M,c) is a conformal *spin* manifold and the aim of this section is to present a spinorial analogue of the first prolongation of the conformal frame bundle, being a first prolongation of the conformal spin bundle $\mathcal{Q}^0_+$. We again first consider the flat model for conformal spin structures in signature (p,q). To this end, let $\widetilde{B^+}, \widetilde{P^+}, \widetilde{B^+_0}, \widetilde{B^+_1}$ denote the preimages of the groups (2.21) under $\lambda : Spin^+(p+1,q+1) \to SO^+(p+1,q+1)$. In particular, $\widetilde{B}^+ = Spin^+(p+1,q+1)$ and $\widetilde{B_0}^+ \cong CSpin^+(p,q)$. The Lie algebras of these groups are isomorphic to those from the corresponding groups (2.21). Here, $\widetilde{P}^+$ is a parabolic subgroup with respect to the $|1|$-grading of $\mathfrak{so}(p+1,q+1)$ and it is isomorphic to $CSpin^+(p,q) \ltimes \mathbb{R}^n$. The pair $(\widetilde{B}^+, \widetilde{P}^+)$ is the flat model of conformal spin geometry, and we have $\widetilde{B}^+/\widetilde{P}^+ \cong B^+/P^+ \cong \widehat{Q}^{p,q}$. Therefore, the Cartan geometry $(\widetilde{B} \to \widetilde{B}/\widetilde{P} \cong \widehat{Q}^{p,q}, \widetilde{\omega}^{MC})$ can be viewed as the conformally flat space $\widehat{C}^{p,q}$ equipped with a natural conformal spin structure (cf. [Lei07]).

Remark 2.31 We view $CSpin^+(p,q)$ as a subgroup of $Spin^+(p+1,q+1)$ because there is a uniquely defined embedding (cf. [Lei01], chapter 2) $i_{cs} : CSpin^+(p,q) \hookrightarrow Spin^+(p+1,q+1)$ such that the following diagram commutes:

$$\begin{tikzcd} CSpin(p,q) \arrow[r, "i_{cs}"] \arrow[d, "\lambda^0"'] & Spin^+(p+1,q+1) \arrow[d, "\lambda"] \\ CO^+(p,q) \arrow[r, "i_c"'] & SO^+(p+1,q+1) \end{tikzcd}$$

Given a conformal spin structure $(\mathcal{Q}^0_+, \widetilde{\pi}^0, M; CSpin^+(p,q))$ of (M,c), the first prolongation is defined to be the set

$$\mathcal{Q}^1_+ := \{H \mid H \subset T_u\mathcal{Q}^0_+, u \in \mathcal{Q}^0_+, df^0_u(H) \in \mathcal{P}^1_+\},$$

and it is equipped with a natural $\widetilde{P}^+$-action, given by

$$H \cdot \widetilde{p} := (df^0_{u\cdot\widetilde{b}_0})^{-1}\left(df^0_u(H)\cdot\lambda(\widetilde{p})\right) \subset T_{u\cdot\widetilde{b}_0}\mathcal{Q}^0_+,$$

where $H \in \mathcal{Q}^1_+$, $H \subset T_u\mathcal{Q}^0_+$, and $\widetilde{p} = (\widetilde{b}_0, \widetilde{b}_1) \in \widetilde{P}^+ = \widetilde{B}^+_0 \ltimes \widetilde{B}^+_1$. Moreover, $\mathcal{Q}^1_+$ comes together with natural projections

$$\widetilde{\pi}^+ : \mathcal{Q}^1_+ \to \mathcal{Q}^0_+,\ H \mapsto u \text{ if } H \subset T_u\mathcal{Q}^0_+,$$
$$\widetilde{\pi}^1 : \mathcal{Q}^1_+ \to M,\ \widetilde{\pi}^1 := \widetilde{\pi}^0 \circ \widetilde{\pi}^+,$$

and a map

$$f^1 : \mathcal{Q}^1_+ \ni H \mapsto df^0(H) \in \mathcal{P}^1_+.$$

Lemma 2.32 ([BJ10], section 2.6) *The above definitions give rise to a $\widetilde{P}^+$-principal bundle $(\mathcal{Q}^1_+, \widetilde{\pi}^1, M; \widetilde{P}^+)$ over M. Moreover, f^1 is a double covering and $(\mathcal{Q}^1_+, f^1)$ is a λ-reduction of $\mathcal{P}^1_+$.*

Clearly, Every Cartan connection $\omega \in \Omega^1(\mathcal{P}^1_+, \mathfrak{b})$ on $\mathcal{P}^1_+$ uniquely defines a Cartan connection $\widetilde{\omega} \in \Omega^1(\mathcal{Q}^1_+, \mathfrak{spin}(p+1, q+1))$ by

$$\widetilde{\omega} := \lambda_*^{-1} \circ \omega \circ df^1. \tag{2.31}$$

In particular, the normal conformal Cartan connection ω^{nc} on $\mathcal{P}^1_+$ defines in this way the normal conformal spin connection $\widetilde{\omega}^{nc}$ on $\mathcal{Q}^1_+$. Note that this construction is completely analogous to the metric setting for $\mathcal{Q}^g_+$. In summary, the first prolongation in the conformal spin setting yields for every $g \in c$ the following embeddings and double coverings:

$$\boxed{\begin{array}{llllll} (\mathcal{Q}^g_+; Spin^+(p,q)) & \hookrightarrow & (\mathcal{Q}^0_+; CSpin^+(p,q)) & \overset{\text{1. Prol.}}{\leftrightarrow} & (\mathcal{Q}^1_+, \widetilde{\omega}^{nc}) \text{ of type} & (\widetilde{B}, \widetilde{P}) \\ f^g \downarrow & & f^0 \downarrow & & f^1 \downarrow & \lambda \downarrow \\ (\mathcal{P}^g_+; SO^+(p,q)) & \hookrightarrow & (\mathcal{P}^0_+; CO^+(p,q)) & \overset{\text{1. Prol.}}{\leftrightarrow} & (\mathcal{P}^1_+, \omega^{nc}) \text{ of type} & (B, P) \end{array}}$$

In analogy to Remark 2.24, fixing $g \in c$ defines a $\widetilde{B^+_0} \cong CSpin^+(p,q)$ equivariant section $\widetilde{\eta}^g : \mathcal{Q}^0_+ \to \mathcal{Q}^1_+$, given by $\widetilde{\eta}^g(q) := \ker\ (\widetilde{\omega}^g_q) \in \mathcal{Q}^1_+$, where $\widetilde{\omega}^g \in \Omega^1(\mathcal{Q}^g_+, \mathfrak{spin}(p,q))$ is canonically extended to $\mathcal{Q}^0_+$. Moreover, the choice of g yields a $Spin^+(p,q) \hookrightarrow CSpin^+(p,q)$ reduction of $(\mathcal{Q}^0_+, f^0)$ to $(\mathcal{Q}^g_+, f^g)$ with reduction map $\widetilde{\iota}^g : \mathcal{Q}^g_+ \hookrightarrow \mathcal{Q}^0_+$ which covers ι^g. The composition then leads to a $Spin^+(p,q)$-equivariant section

$$\widetilde{\sigma}^g : \mathcal{Q}^g_+ \to \mathcal{Q}^1_+,\ \widetilde{\sigma}^g = \widetilde{\eta}^g \circ \widetilde{\iota}^g, \tag{2.32}$$

which is a $i_{cs} : Spin^+(p,q) \to Spin^+(p+1, q+1)$-reduction of $\mathcal{Q}^1_+$ to $\mathcal{Q}^g_+$. By construction, we have that $f^1 \circ \widetilde{\sigma}^g = \sigma^g \circ f^g$, yielding the following commuting diagram:

$$\begin{tikzcd} \mathcal{Q}^g_+ \arrow[r, "\widetilde{\sigma}^g"] \arrow[d, "f^g"'] & \mathcal{Q}^1_+ \arrow[d, "f^1"] \\ \mathcal{P}^g_+ \arrow[r, "\sigma^g"'] & \mathcal{P}^1_+ \end{tikzcd}$$

2.8 Spin tractor bundles

Let (M, c) be a conformal spin manifold. Via the real or complex standard spinor representation $\rho : Spin^+(p+1, q+1) \to GL(\Delta_{p+1,q+1})$ we obtain an associated tractor bundle,

$$\mathcal{S}(M) := \mathcal{Q}^1_+ \times_{\widetilde{P}^+} \Delta_{p+1,q+1},$$

called the (real or complex) spin tractor bundle of (M, c). In complete analogy to the metric case from section 2.3, one can naturally associate to this bundle a triple

$$(\langle \cdot, \cdot \rangle_{\mathcal{S}}, cl, \nabla^{nc}).$$

Here, ∇^{nc} is induced by $\widetilde{\omega}^{nc}$ as explained for general Cartan geometries[10], $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ is defined using the $Spin^+(p+1, q+1)$-invariance of $\langle \cdot, \cdot \rangle_{\Delta_{p+1,q+1}}$, and the identification of $\mathcal{T}(M)$ with a bundle associated to $\mathcal{Q}^1_+$ gives rise to the (pointwise) Clifford multiplication $X \cdot \psi$ of sections

[10] We use the same notation for this covariant derivative as for the covariant derivative on $\mathcal{T}(M)$ when no confusion is likely to occur.

$X \in \Gamma(\mathcal{T}(M))$ and spinor fields $\psi \in \Gamma(\mathcal{S}(M))$. Compatibility conditions analogous to (2.4) hold in this case for $X, Y \in \Gamma(\mathcal{T}(M))$, $\phi, \psi \in \Gamma(\mathcal{S}(M))$ and $Z \in \mathfrak{X}(M)$:

$$\begin{aligned}
(X \cdot Y + Y \cdot X) \cdot \phi &= -2\langle X, Y\rangle_{\mathcal{T}} \cdot \phi, \\
\langle X \cdot \phi, \psi\rangle_{\mathcal{S}} &= (-1)^{p+2}\langle \phi, X \cdot \psi\rangle_{\mathcal{S}}, \\
Z\langle \phi, \psi\rangle_{\mathcal{S}} &= \langle \nabla^{nc}_Z \phi, \psi\rangle_{\mathcal{S}} + \langle \phi, \nabla^{nc}_Z \psi\rangle_{\mathcal{S}}, \\
\nabla^{nc}_Z (Y \cdot \phi) &= (\nabla^{nc}_Z Y) \cdot \phi + Y \cdot \nabla^{nc}_Z \phi.
\end{aligned}$$

In analogy to the treatment from section 2.3, one sees that if half-spinors exist, the spin tractor bundle decomposes into $\mathcal{S}(M) = \mathcal{S}^+(M) \oplus \mathcal{S}^-(M)$, and ∇^{nc} respects this decomposition as follows from the local formula (2.2), applied to our case.

Fixing $g \in c$ yields a reduction of $\mathcal{Q}^1_+$ to $\mathcal{Q}^g_+$, leading to an isomorphism $\mathcal{S}(M) \cong \mathcal{Q}^g_+ \times_{Spin^+(p,q)} \Delta_{p+1,q+1}$. This allows a decomposition of $\mathcal{S}(M)$ wrt. the metric $g \in c$: Let $\overline{\mathcal{Q}^1_+}$ denote the enlarged $Spin^+(p+1, q+1)$-principal bundle. As $\mathcal{S}(M) \cong \overline{\mathcal{Q}^1_+} \times_{Spin^+(p+1,q+1)} \Delta_{p+1,q+1}$, we may use g to identify

$$\begin{aligned}
Q^g_+ \times_{\rho \circ i_{cs}} \Delta_{p+1,q+1} &\cong \mathcal{S}(M), \\
[l, v] &\mapsto [[\widetilde{\sigma}^g(l), e], v],
\end{aligned}$$

where $i_{cs} : Spin^+(p,q) \hookrightarrow Spin^+(p+1, q+1)$ denotes the natural inclusion, and $e \in Spin^+(p+1, q+1)$ is the neutral element. The decomposition (1.11) of $\Delta_{p+1,q+1}$ induces projections $proj^g_\pm : \mathcal{S}(M) \to \overline{\mathcal{Q}^1_+} \times_{Spin^+(p,q)} Ann(e_\pm), [[\sigma^g(l), e], v] \mapsto \left[[\sigma^g(l), e], proj_{Ann(e_\pm)} v\right]$ and a vector bundle isomorphism

$$\begin{aligned}
\widetilde{\Phi}^g : \mathcal{S}(M) &\to S^g(M) \oplus S^g(M) = \mathcal{S}(M)_g, \\
[[\sigma^g(l), e], e_- w + e_+ w] &\mapsto [l, \zeta(e_+ w)] + [l, \chi(e_- w)],
\end{aligned} \tag{2.33}$$

which we call the g-metric representation of $\mathcal{S}(M)$.

We seek for a description of $\nabla^{nc} : \Gamma(\mathcal{S}(M)) \to \Gamma(T^*M \otimes \mathcal{S}(M))$ wrt. this identification. To this end, let $\mathcal{W}$ be one of the bundles TM, T^*M or $\mathfrak{so}(TM, g)$. Analogously to (2.22), $\mathcal{W}$ acts on $\mathcal{S}(M)$ via

$$\widetilde{\rho}^g(\Theta)t := \left[\widetilde{u}, \rho_* \left(\lambda_*^{-1}\left([u]^{-1}\Theta\right)\right)[\widetilde{u}]^{-1} t\right] \in \mathcal{S}_x(M), \tag{2.34}$$

where $\Theta \in \mathcal{W}_x, t \in \mathcal{S}_x(M), \widetilde{u} \in \mathcal{Q}^g_x$ and $u = f^g(\widetilde{u}) \in \mathcal{P}^g_x$. The spinorial analogue of Lemma 2.26 then reads as follows:

Lemma 2.33 ([BJ10]) *The covariant derivative $\nabla^{nc} : \Gamma(\mathcal{S}(M)) \to \Gamma(T^*M \otimes \mathcal{S}(M))$ and its curvature are wrt. a metric $g \in c$ given by*

$$\begin{aligned}
\nabla^{nc}_X &= \nabla^{S^g}_X + \widetilde{\rho}^g(X) + \widetilde{\rho}^g\left(K^g(X)\right), \\
R^{nc}(X, Y) &= \widetilde{\rho}^g(W^g(X, Y)) - \widetilde{\rho}^g(C^g(X, Y)).
\end{aligned} \tag{2.35}$$

We give a more explicit description:

Proposition 2.34 *The g-metric representation of the covariant derivative* ∇^{nc} *on* $\mathcal{S}(M)$*, i.e. the map* $\widetilde{\Phi}^g \circ \nabla^{nc} \circ \left(\widetilde{\Phi}^g\right)^{-1}$ *is given by*

$$\nabla^{nc}_X \begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} \nabla^{S^g}_X & -X\cdot \\ \frac{1}{2}K^g(X)\cdot & \nabla^{S^g}_X \end{pmatrix} \begin{pmatrix} \varphi \\ \phi \end{pmatrix}.$$

Proof. It is known from [Fis13] that the inverse of $\lambda_* : \mathfrak{spin}(p+1,q+1) \to \mathfrak{so}(p+1,q+1)$ applied to $x \in \mathfrak{b}_{-1} \cong \mathbb{R}^{p,q}$ and $z \in \mathfrak{b}_1 \cong (\mathbb{R}^{p,q})^*$ is

$$\lambda_*^{-1}(x) = -\frac{1}{2}x \cdot e_+,$$
$$\lambda_*^{-1}(z) = +\frac{1}{2}z \cdot e_-.$$

Let $X = [u,x] \in TM \cong \mathcal{P}^g \times_{Ad} \mathfrak{b}_{-1}$ for some $u \in \mathcal{P}^g_y$ and $x \in \mathfrak{b}_{-1}$. Then there is $z \in T^*M \cong \mathcal{P}^g \times_{Ad} \mathfrak{b}_1$ with $K^g(X) = [u,z]$. Let $\begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in \Gamma(S^g \oplus S^g)$. We find $w \in \Delta_{p+1,q+1}$ such that $\varphi(y) = [\widetilde{u}, \zeta(e_+ \cdot w)]$ and $\phi(y) = [\widetilde{u}, \chi(e_- \cdot w)]$, where $f^g(\widetilde{u}) = u$, i.e. we have for $\psi := \left(\widetilde{\Phi}^g\right)^{-1}\left(\begin{pmatrix} \varphi \\ \phi \end{pmatrix}\right)$ that $\psi(y) = [[\widetilde{\sigma}^g(\widetilde{u}), e], e_- \cdot w + e_+ \cdot w]$. It follows that

$$\widetilde{\rho}^g(X)\psi(y) = \left[\widetilde{u}, -\frac{1}{2}x \cdot e_+ \cdot (e_- \cdot w + e_+ \cdot w)\right] = \left[\widetilde{u}, -\frac{1}{2}x \cdot e_+ \cdot e_- \cdot w\right],$$
$$\widetilde{\rho}^g(K^g(X))\psi(y) = \left[\widetilde{u}, \frac{1}{2}z \cdot e_- \cdot (e_- \cdot w + e_+ \cdot w)\right] = \left[\widetilde{u}, \frac{1}{2}z \cdot e_- \cdot e_+ \cdot w\right].$$

Consequently, $\widetilde{\rho}^g(X) + \widetilde{\rho}^g(K^g(X)) = \left[\widetilde{u}, -\frac{1}{2} \cdot (e_+ + e_-) \cdot (e_- \cdot x \cdot w - e_+ \cdot z \cdot w)\right]$, and thus

$$\widetilde{\Phi}^g((\widetilde{\rho}^g(X) + \widetilde{\rho}^g(K^g(X)))\psi) = \begin{pmatrix} [\widetilde{u}, -\frac{1}{2} \cdot \chi(e_- \cdot e_+ \cdot e_- \cdot x \cdot w)] \\ [\widetilde{u}, \frac{1}{2}\chi(e_- \cdot e_+ \cdot z \cdot w)] \end{pmatrix} = \begin{pmatrix} [\widetilde{u}, -x \cdot \chi(e_- \cdot w)] \\ [\widetilde{u}, \frac{1}{2}z \cdot \zeta(e_+ \cdot w)] \end{pmatrix}$$
$$= \begin{pmatrix} -X \cdot \phi \\ \frac{1}{2} \cdot K^g(X) \cdot \varphi \end{pmatrix}.$$

From this and Lemma 2.33 the desired formula follows immediately. □

The spinorial analogue of Lemma 2.30 reads as follows:

Lemma 2.35 ([Fis13]) *Let* $\psi \in \Gamma(\mathcal{S}(M))$ *be a spin tractor on a conformal manifold* (M,c)*. Fix* $g \in c$ *and* $\widetilde{g} = e^{2\sigma}g \in c$ *and let* $\varphi = \widetilde{\Phi}^g(proj^g_+\psi)$ *and* $\phi = \widetilde{\Phi}^{\widetilde{g}}(proj^{\widetilde{g}}_+\psi)$ *denote the spinor fields corresponding to* ψ *with respect to* g *and* $\widetilde{g}$*. Then these spinors are related by*

$$\phi = e^{\frac{\sigma}{2}}\widetilde{\varphi},$$

and their Dirac operators satisfy

$$D^{\widetilde{g}}\phi = e^{-\frac{\sigma}{2}}\widetilde{\left(\frac{n}{2} \cdot grad^g\sigma \cdot \varphi + D^g\varphi\right)}.$$

3 Twistor Spinors and Conformal Holonomy

Throughout this whole chapter, let (M,c) denote a space-and time oriented conformal spin manifold with $g \in c$ a metric in the conformal class. The aim of this chapter is the proof of a (partial) classification result for pseudo-Riemannian geometries admitting twistor spinors in arbitrary signature. In the Riemannian case this problem can be viewed as solved:

Theorem 3.1 ([BFGK91]) *Let (M^n,g) be a Riemannian spin manifold of dimension ≥ 3 admitting a nontrivial twistor spinor $\varphi \in \Gamma(S^g)$. Then the following hold:*

1. *The zero set $zero(\varphi)$ is a discrete subset of M and $(M\backslash zero(\varphi), \widetilde{g} = \frac{1}{|\varphi|^4}g)$ is an Einstein manifold of non-negative scalar curvature $scal^{\widetilde{g}}$. If $scal^{\widetilde{g}} = 0$, then $\frac{1}{|\varphi|}\widetilde{\varphi}$ is parallel on $(M\backslash zero(\varphi), \widetilde{g})$. This is the case if $zero(\varphi) \neq \varnothing$. In case $scal^{\widetilde{g}} > 0$, the spinor φ has no zeroes and $\frac{1}{|\varphi|}\widetilde{\varphi}$ is a sum of two Killing spinors.*
2. *Let (M,g) be a compact Riemannian spin manifold. Then there is a rescaled metric $\widetilde{g}$ of constant scalar curvature such that under the canonical map$\widetilde{\ }: S^g \to S^{\widetilde{g}}$ twistor spinors precisely correspond to (linear combinations of) Killing spinors on $(M,\widetilde{g})$.*

There are further important classification results in the Lorentzian case ([BL04, Lei01, Lei07]), which we are going to recall in this chapter, and some results about twistor spinors in low dimensions (cf. [DW07, HS11a, HS11b]). Apart from this, the classification problem for twistor spinors is widely open. Clearly, every parallel spinor on (M,g) is a twistor spinor. Geometries admitting parallel spinors have been widely studied. In general, if $\varphi \in \Gamma(S^g)$ is a parallel spinor, we have that (cf. [Bau81])

$$Ric(X)\cdot\varphi = 0 \ \forall X \in \mathfrak{X}(M). \tag{3.1}$$

In particular, (M,g) is Ricci-isotropic, i.e. the image of $Ric^g : TM \to TM$ is totally lightlike.

Theorem 3.2 ([Kat99]) *Let $(M^{p,q},g)$ be a simply-connected, non locally symmetric, irreducible pseudo-Riemannian spin manifold of dimension $n = p+q$ and let K be the dimension of the space of parallel spinors in $\Gamma(S^g_{\mathbb{C}})$ of M. Then $K > 0$ iff the holonomy representation of (M,g) is (up to conjugation in $SO(p,q)$) one of the following[1]:*

1. $SU(r,s) \subset SO(2r,2s), n = 2(r+s), p = 2r \Rightarrow K = 2$
2. $Sp(r,s) \subset SO(4r,4s), n = 4(r+s), p = 4r \Rightarrow K = r+s+1$
3. $G_2 \subset SO(7), n = 7, p = 0 \Rightarrow K = 1$

[1] For definitions of the groups appearing in this Theorem and realizations as subgroups of $SO(p,q)$, we refer to [Kat99, Bau09] and references therein.

4. $G_{2,2} \subset SO(4,3), n = 7, p = 4 \Rightarrow K = 1$

5. $G_2^{\mathbb{C}} \subset SO(7,7)$*,* $n = 14, p = 7 \Rightarrow K = 2$

6. $Spin(7) \subset SO(8), n = 8, p = 0 \Rightarrow K = 1$

7. $Spin^+(4,3) \subset SO(4,4), n = 8, p = 4 \Rightarrow K = 1$

8. $Spin(7)^{\mathbb{C}} \subset SO(8,8), n = 16, p = 8 \Rightarrow K = 1$

Moreover, one can give the chiral and causal type of the parallel spinors.

Furthermore, [Kat99] discusses parallel pure spinors in split signature and [BLL12] describes full holonomy groups of Lorentzian manifolds with parallel null spinor. Our first goal in the classification of geometries admitting twistor spinors is to distinguish those twistor spinors which are parallel wrt. some metric in the conformal class. The further study of such geometries is then reduced to the investigation of pseudo-Riemannian geometries admitting parallel spinors which is not the main purpose of this thesis. We are more interested in conformal admitting true twistor spinors, i.e. twistor spinors which cannot be rescaled to parallel spinors (not even locally).
To this end, we show how the classification problem for twistor spinors is related to the classification problem for conformal holonomy groups $Hol(M,c)$. We then prove a classification result for reducible conformal holonomy representations which applied to twistor spinors precisely distinguishes those which are equivalent to parallel spinors. Finally, we give a partial classification result for manifolds admitting true twistor spinors in terms of conformal holonomy.

3.1 Twistor spinors and parallel tractors

We have introduced twistor spinors $\varphi \in \ker P^g$ wrt. a metric g and studied their elementary properties in the previous chapter. Propositions 2.34 and 2.12 immediately yield an important reinterpretation of twistor spinors in terms of conformal Cartan geometry:

Theorem 3.3 *Let (M,c) be a connected, space- and time-oriented conformal spin manifold of dimension $n \geq 3$. For any metric $g \in c$, the vector spaces of twistor spinors $ker\ P^g \subset \Gamma(S^g)$ and parallel sections in $\Gamma(\mathcal{S}(M))$ are naturally isomorphic via*

$$\widetilde{\Phi}_{ker\ P^g} : ker\ P^g \to \Gamma(S^g(M) \oplus S^g(M)) \overset{\left(\widetilde{\Phi}^g\right)^{-1}}{\cong} \Gamma(\mathcal{S}(M))$$
$$\varphi \mapsto \begin{pmatrix} \varphi \\ -\frac{1}{n} D^g \varphi \end{pmatrix} \overset{\left(\widetilde{\Phi}^g\right)^{-1}}{\mapsto} \psi \in Par(\mathcal{S}(M), \nabla^{nc}),$$

i.e. a spin tractor $\psi \in \Gamma(\mathcal{S}(M))$ is parallel iff for one (and hence for all) $g \in c$, it holds that $\varphi := \widetilde{\Phi}^g\,(proj^g_+ \psi) \in ker\ P^g$, and in this case one has $\widetilde{\Phi}^g\,(proj^g_- \psi) = -\frac{1}{n} D^g \varphi$.

We will henceforth always identify for a fixed metric the spaces $Par(\mathcal{S}(M), \nabla^{nc})$ and $\ker P^g$ by means of Theorem 3.3. There is a canonical way of associating other parallel tractors to a twistor spinor which we shall exploit in this section. First, let $\varphi_{1,2} \in \Gamma(S^g)$ and $\psi_{1,2} \in \Gamma(\mathcal{S}(M))$ be arbitrary spinor fields. The algebraic construction from section 1.4

can be made global by defining the following forms $\alpha^k_{\psi_1,\psi_2} \in \Omega^k_{\mathcal{T}}(M)$ and $\alpha^k_{\varphi_1,\varphi_2} \in \Omega^k(M)$ for every $k \in \mathbb{N}$ (as done in [Lei05, Lei09]):

$$\begin{array}{rclllll} \langle \alpha^k_{\psi_1,\psi_2}, \alpha \rangle_{\mathcal{T}} & := & d_{k,p+1} & (\langle \alpha \cdot \psi_1, \psi_2 \rangle_{\mathcal{S}}) & \forall & \alpha & \in & \Omega^k_{\mathcal{T}}(M), \\ g(\alpha^k_{\varphi_1,\varphi_2}, \alpha) & := & d_{k,p} & (\langle \alpha \cdot \varphi_1, \varphi_2 \rangle_{S^g}) & \forall & \alpha & \in & \Omega^k(M). \end{array} \tag{3.2}$$

Remark 3.4 The maps $d : \mathbb{K} \to \mathbb{K}$ are to be interpreted as in section 1.4. We set $\alpha^k_\psi := \alpha^k_{\psi,\psi}$. Furthermore, if $\psi \in \Gamma(\mathcal{S}(M))$ is locally given by $\psi(x) = [\widetilde{u}(x), \eta(x)]$ for a local section $\widetilde{u} : U \to \mathcal{Q}^1_+$ and a smooth map $\eta : U \to \Delta_{p+1,q+1}$, then α^k_ψ is locally given by $\alpha^k_\psi(x) = [u(x), \alpha^k_{\eta(x)}]$, where $u = f^1 \circ \widetilde{u}$. Note moreover that the algebraic properties of algebraic Dirac forms from section 1.4 directly translate into corresponding pointwise properties of α^k_ψ and α^k_φ.

Proposition 3.5 *Let $(M^{p,q}, c)$ be a conformal spin manifold and let $\psi_{1,2} \in Par(\mathcal{S}(M), \nabla^{nc})$. Then for each $k \in \mathbb{N}$, the tractor k–form $\alpha^k_{\psi_1,\psi_2}$ is parallel wrt. ∇^{nc}.*

Proof. Let $X \in \mathfrak{X}(M)$ be arbitrary. We have for all $\alpha \in \Omega^k_{\mathcal{T}}(M)$ that

$$\begin{aligned} X\langle \alpha^k_{\psi_1,\psi_2}, \alpha \rangle_{\mathcal{T}} &= d_{k,p+1} \left(X \langle \alpha \cdot \psi_1, \psi_2 \rangle_{\mathcal{S}} \right) \\ &= d_{k,p+1} \left(\langle (\nabla^{nc}_X \alpha) \cdot \psi_1, \psi_2 \rangle_{\mathcal{S}} + \langle \alpha \cdot \nabla^{nc}_X \psi_1, \psi_2 \rangle_{\mathcal{S}} + \langle \alpha \cdot \psi_1, \nabla^{nc}_X \psi_2 \rangle_{\mathcal{S}} \right) \\ &= d_{k,p+1} \left(\langle (\nabla^{nc}_X \alpha) \cdot \psi_1, \psi_2 \rangle_{\mathcal{S}} \right), \end{aligned}$$

where the constant is nonzero. Consequently,

$$\langle \nabla^{nc}_X \alpha^k_{\psi_1,\psi_2}, \alpha \rangle_{\mathcal{T}} + \langle \alpha^k_{\psi_1,\psi_2}, \nabla^{nc}_X \alpha \rangle_{\mathcal{T}} = d_{k,p+1} \left(\langle (\nabla^{nc}_X \alpha) \cdot \psi_1, \psi_2 \rangle_{\mathcal{S}} \right). \tag{3.3}$$

We now fix $x \in M$. The fact that $\nabla^{nc} \langle \cdot, \cdot \rangle_{\mathcal{T}} = 0$ allows the choice of a pseudo-orthonormal basis frame $(s_0 ..., s_{n+1})$ in $\mathcal{T}(M)$ around x which is parallel in x. It follows that also $\nabla^{nc}_X (s^\flat_{i_1} \wedge ... \wedge s^\flat_{i_k}) = 0$ at x. Then, (3.3) reads $\langle \nabla^{nc}_X \alpha^k_{\psi_1,\psi_2}, s^\flat_{i_1} \wedge ... \wedge s^\flat_{i_k} \rangle_{\mathcal{T}}(x) = 0$ for all $1 \leq i_1 < ... < i_r \leq n$, and therefore $\nabla^{nc}_X \alpha^k_{\psi_1,\psi_2}(x) = 0$. □

Remark 3.6 A completely analogous proof reveals the well-known fact that on a pseudo-Riemannian spin manifold (M, g) admitting a parallel spinor $\varphi \in \Gamma(S^g)$ the forms α^k_φ are parallel for every $k \in \mathbb{N}$.

In general, we call every parallel tractor $(k+1)$-form $\alpha \in Par(\Lambda^{k+1}_{\mathcal{T}}(M), \nabla^{nc}) \subset \Omega^{k+1}_{\mathcal{T}}(M)$, i.e $\nabla^{nc} \alpha = 0$, a twistor-$(k+1)$-form. Proposition 3.5 motivates us to study general properties of twistor forms. As a direct consequence of Lemma 2.29 we note that for every $g \in c$ the requirement $\nabla^{nc} \alpha = 0$ is equivalent to the following conditions on $\Phi^g_\Lambda(\alpha) = (\alpha_+, \alpha_0, \alpha_\mp, \alpha_-) \in \Omega^k(M) \oplus \Omega^{k+1}(M) \oplus \Omega^{k-1}(M) \oplus \Omega^k(M)$:

$$\begin{aligned} \nabla^g_X \alpha_+ - X \lrcorner \alpha_0 - X^\flat \wedge \alpha_\mp &= 0, \\ -K^g(X)^\flat \wedge \alpha_+ + \nabla^g_X \alpha_0 + X^\flat \wedge \alpha_- &= 0, \\ -K^g(X) \lrcorner \alpha_+ + \nabla^g_X \alpha_\mp - X \lrcorner \alpha_- &= 0, \\ K^g(X) \lrcorner \alpha_0 - K^g(X)^\flat \wedge \alpha_\mp + \nabla^g_X \alpha_- &= 0. \end{aligned} \tag{3.4}$$

As it turns out, all these forms can be expressed in terms of α_+ only, namely a straightforward calculation as carried out in [Lei05] yields

$$\begin{aligned} d\alpha_+ &= (k+1)\alpha_0, \\ d^*\alpha_+ &= -(n-k+1)\alpha_\mp, \\ \alpha_- &= \Box_k \alpha_+, \end{aligned} \tag{3.5}$$

whereby we have set

$$\Box_k := \begin{cases} \frac{1}{n-2k}\left(-\frac{scal^g}{2(n-1)} + \nabla^*\nabla\right), & n \neq 2k, \\ \frac{1}{n}\left(\frac{1}{k+1}(d^*d + dd^*) + \sum_{i=1}^n \epsilon_i \left(s_i \lrcorner (K^g(s_i)^\flat \wedge \cdot) - s_i^\flat \wedge (K^g(s_i) \lrcorner \cdot)\right)\right), & n = 2k. \end{cases}$$

Here, $s = (s_1, ..., s_n)$ is a local section of $\mathcal{P}^g$ and ∇^* denotes the formal adjoint of $\nabla = \nabla^g$. With the so derived expressions for the components of a normal twistor form with respect to a metric, we can can express the conformally covariant equations (3.4) in terms of α_+ only:

$$\begin{aligned} \nabla^g_X \alpha_+ - \frac{1}{k+1} X \lrcorner d\alpha_+ + \frac{1}{n-k+1} X^\flat \wedge d^*\alpha_+ &= 0, \\ -K^g(X)^\flat \wedge \alpha_+ + \frac{1}{k+1}\nabla^g_X d\alpha_+ + X^\flat \wedge \Box_k \alpha_+ &= 0, \\ K^g(X)\lrcorner\alpha_+ + \frac{1}{n-k+1}\nabla^g_X d^*\alpha_+ + X \lrcorner \Box_k \alpha_+ &= 0, \\ \frac{1}{k+1}K^g(X)\lrcorner d\alpha_+ + \frac{1}{n-k+1}K(X)^\flat \wedge d^*\alpha_+ + \nabla^g_X \Box_k \alpha_+ &= 0. \end{aligned} \tag{3.6}$$

A differential form $\alpha_+ \in \Omega^k(M)$ is called a normal conformal Killing k-form (or shorty, a nc-Killing form), if it satisfies the equations (3.6). We denote the set of these forms by $\Omega^k_{nc,g}(M)$. Only considering the first equation in (3.6) leads to conformal Killing forms as studied in great detail in [Sem01]. A conformal Killing form which is closed for some metric $g \in c$ is called a Killing form for (M, g). In summary, one has a relation between twistor $(k+1)$-forms and nc-Killing k–forms wrt. $g \in c$ which can be viewed as the analogue of Theorem 3.3 on the space of forms.

Theorem 3.7 *Let $(M^{p,q}, c)$ be a conformal manifold. The choice of $g \in c$ leads to a natural isomorphism*

$$\begin{aligned} Par\left(\Lambda^{k+1}_{\mathcal{T}}(M), \nabla^{nc}\right) &\to \Omega^k_{nc,g}(M), \\ \alpha &\mapsto proj^g_{\Lambda,+}(\alpha), \end{aligned}$$

where the inverse is given by $\alpha_+ \mapsto \left(\Phi^g_\Lambda\right)^{-1}\left(\alpha_+, \frac{1}{k+1}d\alpha_+, -\frac{1}{n-k+1}d^*\alpha_+, \Box_k\alpha_+\right)$.

The existence of nontrivial twistor forms has many interesting implications on the (local) geometry of M. This is elaborated on in full detail in [Lei05] and [Lei07]. We later need the following result: We say that a (nontrivial) form $\sigma \in \Lambda^r_{p,q}$ is totally lightlike, if it is decomposable, i.e. $\sigma = \sigma_1 \wedge ... \wedge \sigma_r$, and the σ_i are mutually orthogonal and can be chosen to be lightlike[2].

Lemma 3.8 ([Lei05]) *Let $\alpha^{k+1} \in \Omega^{k+1}_{\mathcal{P}^1}(M)$ be a decomposable twistor $(k+1)$-form with totally lightlike nc-Killing form $\alpha^k_+ = proj^g_{\Lambda,+}(\alpha)$[3]. Then there is locally around each point a metric $\widetilde{g} \in c$ with $\nabla^{\widetilde{g}}\widetilde{\alpha}^k_- = 0$.*

[2]Clearly, these properties do not depend on the particular choice of the σ_i

[3]Note that by Lemma 2.30 this requirement does not depend on the choice of $g \in c$

We turn again to twistor spinors. Let $\psi \in Par\left(\Lambda^{k+1}_{\mathcal{T}}(M)\right)$, $g \in c$ and $\varphi := \widetilde{\Phi}^g\left(proj^g_+\psi\right) \in$ ker P^g. It has been shown in [Lei09][4] that there are constants $c^i_{k,p} \neq 0$ for $i = 1, 2$ such that

$$proj^g_{\Lambda,+}\left(\alpha^{k+1}_\psi\right) = c^1_{k,p} \cdot \alpha^k_\varphi \text{ and } proj^g_{\Lambda,-}\left(\alpha^{k+1}_\psi\right) = c^2_{k,p} \cdot \alpha^k_{D^g\varphi}. \tag{3.7}$$

In particular, (3.7) reveals that for every twistor spinor $\varphi \in \ker P^g$, the forms α^k_φ are nc-Killing forms. Together with the conformal transformation behaviour from Lemma 2.30 and 2.35 under a change $\widetilde{g} = e^{2\sigma}g$, this may be visualized in the following commutative diagram:

$$\begin{tikzcd}
\varphi \in \ker P^g \arrow[r, "\left(\widetilde{\Phi}^g \circ proj^g_+\right)^{-1}"] \arrow[d, "\text{nc-Killing}"] & \psi \in Par(\mathcal{S}(M), \nabla^{nc}) \arrow[r, "\widetilde{\Phi}^{\widetilde{g}} \circ proj^{\widetilde{g}}_+"] \arrow[d, "\text{twistor form}"] & e^{\frac{\sigma}{2}}\widetilde{\varphi} \in \ker P^{\widetilde{g}} \arrow[d, "\text{nc-Killing}"] \\
c^1_{k,p} \cdot \alpha^k_\varphi \in \Omega^k_{nc,g}(M) \arrow[r, "\left(proj^g_{\Lambda,+}\right)^{-1}"] & \alpha^{k+1}_\psi \in \Omega^{k+1}_{\mathcal{T}}(M) \arrow[r, "proj^{\widetilde{g}}_{\Lambda,+}"] & c^1_{k,p} \cdot e^{(k+1)\sigma}\alpha^k_\varphi \in \Omega^k_{nc,\widetilde{g}}(M)
\end{tikzcd}$$

Remark 3.9 Consider the special case of a twistor 2-form $\alpha \in \Omega^2_{\mathcal{T}}(M)$. Then (3.6) yields that for $g \in c$, the vector field $V_\alpha := V_{\alpha_+} := \left(proj^g_{\Lambda,+}\alpha\right)^\sharp$ is a normal conformal vector field, i.e. the dual of a nc-Killing 1-form. By Lemma 2.30, V_α does not depend on the choice of $g \in c$. We denote the space of all normal conformal vector fields on (M, g) by $\mathfrak{X}^{nc}(M)$. [Raj06] shows that for a vector field V being normal conformal is equivalent to being conformal, $V \in \mathfrak{X}^c(M)$, and to satisfy in addition that

$$V \lrcorner W^g = 0, \;\; V \lrcorner C^g = 0. \tag{3.8}$$

If $\alpha = \alpha^2_{\psi_1,\psi_2}$ for parallel spin tractors ψ_1, ψ_2, we obtain in this way the conformal vector field V_{φ_1,φ_2} associated to $\varphi_{1,2} := \widetilde{\Phi}^g(proj^g_+\psi_{1,2}) \in \ker P^g$, which is locally given by

$$V_{\varphi_1,\varphi_2} = \sum_{i=1}^n \epsilon_i d_{1,p}\left(\langle s_i \cdot \varphi_1, \varphi_2 \rangle_{S^g}\right) s_i.$$

Properties of V_φ and their relations to Lorentzian conformal structures admitting twistor spinors have been intensively studied in [Lei01, BL04, Lei07], for instance.

3.2 Holonomy reductions imposed by a twistor spinor

Let (M, c) be a conformal spin manifold with first prolongation $f^1 : \mathcal{Q}^1_+ \to \mathcal{P}^1_+$. Denote by $\overline{\mathcal{Q}}^1_+$ and $\overline{\mathcal{P}}^1_+$ the extended bundles with structure group $Spin^+(p+1, q+1)$ and $SO^+(p+1, q+1)$, respectively. f^1 naturally extends to a double covering $\overline{f}^1 : \overline{\mathcal{Q}}^1_+ \to \overline{\mathcal{P}}^1_+$, yielding a $\lambda-$reduction of $\overline{\mathcal{P}}^1_+$ to $\overline{\mathcal{Q}}^1_+$. Further, denote by $\overline{\omega}^{nc} \in \Omega^1\left(\overline{\mathcal{P}}^1_+, \mathfrak{so}(p+1, q+1)\right)$ and $\overline{\overline{\omega}}^{nc} \in \Omega^1\left(\overline{\mathcal{Q}}^1_+, \mathfrak{spin}(p+1, q+1)\right)$ the principal bundle connections naturally induced by the normal conformal Cartan connection ω^{nc}. The following Proposition can be viewed as a conformal analogue of a similar fact in the metric case as known from [Kat99].

[4]In fact, the reference proves (3.7) only for the Lorentzian case, but a generalization to arbitrary signature is straightforward.

Proposition 3.10 *Let $\widetilde{u} \in \overline{\mathcal{Q}}^1_+$ and let $u := \overline{f}^1(\widetilde{u}) \in \overline{\mathcal{P}}^1_+$. Then the associated holonomy groups are related by*

$$Hol_u\left(\overline{\omega}^{nc}\right) = \lambda\left(Hol_{\widetilde{u}}\left(\widetilde{\overline{\omega}}^{nc}\right)\right).$$

Proof. For $x \in M$ and $\gamma : [a,b] \to M$ a loop which closes in x, let $\gamma^*_{\widetilde{u}} : [a,b] \to \overline{\mathcal{Q}}^1_+$ and $\gamma^*_u : [a,b] \to \overline{\mathcal{P}}^1_+$ denote the horizontal lifts with starting points $\widetilde{u}$ and $u = \overline{f}^1(\widetilde{u})$, respectively. We claim that $\overline{f}^1\left(\gamma^*_{\widetilde{u}}\right) = \gamma^*_u$. To this end, we observe that

1. $\pi_{\overline{\mathcal{P}}^1_+}\left(\overline{f}^1\left(\gamma^*_{\widetilde{u}}(t)\right)\right) = \pi_{\overline{\mathcal{Q}}^1_+}\left(\gamma^*_{\widetilde{u}}(t)\right) = \gamma(t)$ for all $t \in [a,b]$,
2. $\overline{f}^1\left(\gamma^*_{\widetilde{u}}(a)\right) = \overline{f}^1(\widetilde{u}) = u$,
3. $\overline{\omega}^{nc}\left(\frac{d}{dt}\overline{f}^1\left(\gamma^*_{\widetilde{u}}\right)(t)\right) = \overline{\omega}^{nc}\left(d\overline{f}^1\left(\frac{d}{dt}\gamma^*_{\widetilde{u}}(t)\right)\right) = \lambda_*\left(\widetilde{\overline{\omega}}^{nc}\left(\frac{d}{dt}\gamma^*_{\widetilde{u}}(t)\right)\right) = 0.$

Consequently, the claim concerning γ^*_u follows from the uniqueness of the horizontal lift. Now we consider parallel displacement P along γ. By definition, we have that

$$P^{\overline{\omega}^{nc}}_\gamma(u) = \gamma^*_u(b) = \overline{f}^1\left(\gamma^*_{\widetilde{u}}(b)\right) = \overline{f}^1\left(P^{\widetilde{\overline{\omega}}^{nc}}(\widetilde{u})\right),$$

i.e. $P^{\overline{\omega}^{nc}}_\gamma \circ \overline{f}^1 = \overline{f}^1 \circ P^{\widetilde{\overline{\omega}}^{nc}}_\gamma$. Consequently,

$$P^{\widetilde{\overline{\omega}}^{nc}}_\gamma(\widetilde{u}) = \widetilde{u} \cdot \widetilde{g} \Leftrightarrow P^{\overline{\omega}^{nc}}_\gamma(u) = \overline{f}^1(\widetilde{u} \cdot \widetilde{g}) = u \cdot \lambda(\widetilde{g}).$$

With the definition of the holonomy groups the claim follows immediately. □

Remark 3.11 The holonomy principle leads to an isomorphism

$$Par\left(\mathcal{S}(M), \nabla^{nc}\right) \overset{\cong}{\to} V_{\widetilde{u}} := \{v \in \Delta_{p+1,q+1} \mid Hol_{\widetilde{u}}\left(\widetilde{\overline{\omega}}^{nc}\right)v = v\}.$$

Moreover, $\mathfrak{hol}_{\widetilde{u}}\left(\widetilde{\overline{\omega}}^{nc}\right) \overset{\text{Prop. 3.10}}{=} \lambda_*^{-1}\mathfrak{hol}_u\left(\overline{\omega}^{nc}\right)$, and if M is simply-connected, we deduce that

$$V_{\widetilde{u}} = V_{\mathfrak{h}_{\widetilde{u}}} := \{v \in \Delta_{p+1,q+1} \mid \left(\lambda_*^{-1}\mathfrak{hol}_u\left(\overline{\omega}^{nc}\right)\right)(v) = 0\}. \tag{3.9}$$

Using that $Hol_x(M,c) \cong Hol_{\widetilde{u}}\left(\widetilde{\overline{\omega}}^{nc}\right)$, where $u \in \overline{\mathcal{P}}^1_x$, this opens up a conceptual way to determine the space of twistor spinors on a given conformal spin manifold: For given (M,c) the space of parallel spin tractors (or twistor spinors when $g \in c$ is fixed) is in bijection to the space of spinors which are fixed by the lift of $Hol(M,c)$ to $Spin^+(p+1,q+1)$, and in case of M being simply-connected this space can be identified -up to conjugation- with the kernel

$$V_{\mathfrak{hol}(M,c)} := \{v \in \Delta_{p+1,q+1} \mid \left(\lambda_*^{-1}\mathfrak{hol}(M,c)\right) \cdot v = 0\}. \tag{3.10}$$

(3.10) indicates a strong relation between the classification problem for geometries admitting twistor spinors and the classification of possible conformal holonomy groups. Unfortunately, there is except from the Riemannian case (cf. [Arm07]) no complete classification of possible conformal holonomy groups available.
However, let us now apply the above strategy in order to determine the space of twistor

spinors to a situation where all possible holonomy groups are known: To state this, note that the standard matrix action of a subgroup $H \subset O(p+1,q+1)$ induces a natural action on the conformal Möbius sphere $Q^{p,q}$, the projectivized null-cone in $\mathbb{R}^{p+1,q+1}$. We call a conformal holonomy group transitive if this action is transitive (cf. [Alt12], chapter 1).

Theorem 3.12 ([Alt12, ASL14]) *Let $H = Hol(M,[g]) \subset O(p+1,q+1)$ be a connected conformal holonomy group for a conformal manifold of signature (p,q), $n = p+q \geq 3$, and assume that H acts transitively on the conformal Möbius sphere. If H acts irreducibly on $\mathbb{R}^{p+1,q+1}$, then it is isomorphic to one of the following:*

1. *$SO^+(p+1,q+1)$ for all p,q*
2. *$SU(k+1,m+1)$ for $p = 2k+1, q = 2m+1$*
3. *$Sp(1)Sp(k+1,m+1)$ for $p = 4k+3, q = 4m+3$*
4. *$Sp(k+1,m+1)$ for $p = 4k+3, q = 4m+3$*
5. *$Spin^+(1,8)$ for $p = q = 7$*
6. *$Spin^+(4,3)$ for $p = q = 3$*
7. *$G_{2,2}$ for $p = 3, q = 2$*

We can completely describe the space of complex twistor spinors $\ker P^g \subset \Gamma(S^g_{\mathbb{C}})$ for these special conformal holonomies:

Theorem 3.13 *Let (M,c) be a simply-connected conformal spin manifold of signature (p,q) with $p+q \geq 3$. Assume that $Hol(M,c)$ acts transitively on $Q^{p,q}$ and irreducibly on $\mathbb{R}^{p+1,q+1}$. Let $g \in c$. Then $N :=\dim \ker P^g > 0$ if and only if the conformal holonomy representation $H := Hol(M,c)$ of M is (up to conjugation in $SO^+(p+1,q+1)$) one in the list given below:*

1. *$H = SU(k+1,m+1)$ for $p = 2k+1, q = 2m+1$. Then $N = 2$. There exists a basis $\{\varphi_1,\varphi_2\}$ of $\ker P^g$ such that $\varphi_1,\varphi_2 \in \Gamma(S^g_+)$ or $\varphi_1,\varphi_2 \in \Gamma(S^g_-)$ if $k+m$ is even, $\varphi_1 \in \Gamma(S^g_+), \varphi_2 \in \Gamma(S^g_-)$ if $k+m$ is odd.*
2. *$H = Sp(k+1,m+1)$ for $p = 4k+3, q = 4m+3$. Then $N = k+m+3$. We have moreover that $\ker P^g \subset \Gamma(S^g_+)$ or $\ker P^g \subset \Gamma(S^g_-)$.*
3. *$H = Spin^+(4,3)$ for $p = q = 3$. Then $N = 1$. There is a nontrivial real twistor half-spinor φ satisfying $\langle \varphi, D^g\varphi\rangle_{S^g} = const. \neq 0$.*
4. *$H = G_{2,2}$ $p = 3, q = 2$. Then $N = 1$ and there is a nontrivial real twistor spinor φ satisfying $\langle \varphi, D^g\varphi\rangle_{S^g} = const. \neq 0$.*

Proof. The proof is a direct application of the procedure we have just described: All one has to do is to go through the list from Theorem 3.12 and determine the kernel $V_{\mathfrak{hol}(M,c)} \subset \Delta^{\mathbb{C}}_{p+1,q+1}$ (cf. (3.10)). This leads to computations which are pure linear algebra, in fact they have already been carried out in the proof of Theorem 3.2 in [Kat99] for the pseudo-Riemannian Berger list when studying pseudo-Riemannian holonomy groups admitting parallel spinors. One finds that

1. $V_{\mathfrak{so}(p+1,q+1)} = \{0\}$.

2. $V_{\mathfrak{su}(k+1,m+1)}$ is 2-dimensional. If $k+m$ is even, we have that $V_{\mathfrak{su}(k+1,m+1)}$ is contained in either $\Delta^{\mathbb{C},+}_{p+1,q+1}$ or $\Delta^{\mathbb{C},-}_{p+1,q+1}$, whereas in case $k+m$ odd, $V_{\mathfrak{su}(k+1,m+1)}$ contains spinors of both chiralities.

3. $V_{\mathfrak{sp}(1)\oplus\mathfrak{sp}(k+1,m+1)} = \{0\}$.

4. $V_{\mathfrak{sp}(k+1,m+1)}$ is $k+m+3$–dimensional and contained in either $\Delta^{\mathbb{C},+}_{p+1,q+1}$ or $\Delta^{\mathbb{C},-}_{p+1,q+1}$.

5. $V_{\mathfrak{spin}(1,8)} = \{0\}$. This case is not covered by [Kat99][5] and verified by a direct straightforward calculation using the realisation of the maximal compact subgroup $Spin(8) \subset Spin(1,8) \subset SO(8,8)$ from [Bry00].

6. $V_{\mathfrak{spin}(4,3)}$ is one-dimensional. There is a real structure on $\Delta^{\mathbb{C}}_{4,4}$ and $V_{\mathfrak{spin}(4,3)}$ contains a nonzero real spinor $\psi \in \Delta^{\mathbb{R},\pm}_{4,4}$ which additionally satisfies $\langle \psi,\psi\rangle_{\Delta^{\mathbb{R}}_{4,4}} \neq 0$.

7. $V_{\mathfrak{g}_{2,2}}$ is one-dimensional. There is a real structure on $\Delta^{\mathbb{C}}_{4,3}$ and $V_{\mathfrak{g}_{2,2}}$ contains a nonzero real spinor $\psi \in \Delta^{\mathbb{R}}_{4,3}$ which additionally satisfies $\langle \psi,\psi\rangle_{\Delta^{\mathbb{R}}_{4,3}} \neq 0$.

Remark 3.11 and the holonomy principle directly translate these algebraic properties of the holonomy representation into the existence of twistor spinors as claimed in the Theorem. The fact that the real twistor spinors $\varphi \in \Gamma(S^g)$ in the last two cases satisfy $\langle \varphi, D^g\varphi\rangle_{S^g} =$ const. $\neq 0$ is proved in [HS11a]. □

Remark 3.14 The geometric meanings of the conformal holonomy groups appearing in Theorem 3.13 are well-understood: Special unitary conformal holonomy means that there is locally a Fefferman spin space over a strictly pseudoconvex spin manifold in the conformal class (cf. [BJ10, Lei07]) for which the existence of global twistor spinors is well-known and can also be derived without using conformal tractor calculus (cf. [Bau99]). Conformal structures with $Hol(M,c) \subset Sp(k+1,m+1)$ were studied in detail in [Alt08]. The models of such manifolds are S^3-bundles over a quaternionic contact manifold equipped with a canonical conformal structure. Moreover, conformal holonomy sitting in $G_{2,2}$ is studied in [HS11a]. These geometries can be equivalently characterized by the existence of a generic 2-distribution on a 5-dimensional manifold. Similarly, the case $Hol(M,c) \subset Spin^+(4,3)$ which is discussed in [HS11a, Bry00] is completely described in terms of a generic 3-distribution on a 6-dimensional manifold M of split signature. We will elaborate on the last two cases in chapter 5 where we also present solutions to non-generic cases, i.e. geometries admitting twistor spinors satisfying $\langle \varphi, D^g\varphi\rangle = 0$, in these signatures.

Remark 3.15 A complete classification of irreducibly acting conformal holonomy groups is hindered by the circumstance that $\mathfrak{hol}(M,c)$ does not satisfy helpful algebraic properties, such as being a Berger algebra as in case of metric holonomy algebras, which could reduce the classification problem to a finite list of possible algebras.

[5]Note that $Spin^+(1,8) \subset SO^+(8,8)$, cf. [Alt12] for more details, does not appear in the pseudo-Riemannian Berger list. It is yet unclear whether it can be realized as conformal holonomy group.

3.3 Reducible conformal holonomy

So far we have only considered irreducible conformal structures admitting twistor spinors. Let us now turn to a special situation where $Hol(M,c)$ acts reducible with a fixed totally lightlike subspace and describe possible local geometries in these cases. To this end, let $\psi \in \Gamma(\mathcal{S}(M))$ be a parallel spin tractor. We set

$$\ker\,\psi(x) := \{v \in \mathcal{T}_x(M) \mid v \cdot \psi(x) = 0\}. \tag{3.11}$$

Performing this for every point yields a totally lightlike distribution $\ker\,\psi \subset \mathcal{T}(M)$. It is moreover parallel wrt. ∇^{nc}, and henceforth of constant rank, since $Y \in \Gamma(\ker\,\psi)$ and $X \in \mathfrak{X}(M)$ implies that $0 = \nabla^{nc}_X(Y \cdot \psi) = (\nabla^{nc}_X Y)\cdot\psi$. In terms of conformal holonomy, this can be equivalently expressed as follows: Let $\widetilde{u} \in \overline{\mathcal{Q}}^1_+$ and $v \in \Delta_{p+1,q+1} \in V_{\widetilde{u}}$ be the $Hol_{\widetilde{u}}(\overline{\widetilde{\omega}}^{nc})$-invariant spinor associated to ψ. It follows that $\ker\,v = \{l \in \mathbb{R}^{p+1,q+1} \mid t \cdot v = 0\} \subset \mathbb{R}^{p+1,q+1}$ is $Hol_{\overline{f}^1(\widetilde{u})}(\overline{\omega}^{nc})$-invariant, and by parallel displacement, this holonomy invariant subspace precisely gives us $\ker\,\psi$. We now consider the case where this construction is nontrivial, i.e. we *assume* that

$$\ker\,\psi \neq \{0\}. \tag{3.12}$$

Consequently, every parallel spin tractor ψ satisfying (3.12), naturally gives rise to a distinguished totally lightlike subspace fixed by the holonomy representation,

$$Hol_x(M,c)\,\ker\,\psi(x) \subset\, \ker\,\psi(x), \tag{3.13}$$

and we want to understand the geometric implications of (3.13).

As a digression, (3.13) motivates us to consider and classify the more general situation of conformal manifolds admitting reducible conformal holonomy[6]. Let us first review some known facts:

Theorem 3.16 ([BJ10], Prop. 2.3.2) *Let (M,c) be a simply-connected conformal manifold of dimension ≥ 3 and suppose that there is a $Hol_x(M,c)$-invariant 1-dimensional subspace $V^1 \subset \mathcal{T}_x(M) \cong \mathbb{R}^{p+1,q+1}$. Then we distinguish the following cases:*

- *V^1 is spacelike: There exists an Einstein metric $g \in c_{|\widetilde{M}}$ on an open dense subset $\widetilde{M} \subset M$ with $scal^g < 0$.*
- *V^1 is timelike: There is an Einstein metric $g \in c_{|\widetilde{M}}$ on an open dense subset $\widetilde{M} \subset M$ with $scal^g > 0$.*
- *V^1 is lightlike: There is a Ricci-flat metric in c at least outside a singular set.*

Conversely, if there is an Einstein metric g in the conformal class, then there exists a $Hol(M,c)$-invariant vector $v \in \mathbb{R}^{p+1,q+1}\backslash\{0\}$ with $sgn\ \langle v,v\rangle_{p+1,q+1} = -sgn\ scal^g$.

In view of Theorem 3.16, we review an ambient metric construction for Einstein spaces which relates their conformal holonomy to metric holonomy, following [BJ10, Lei01]: Let

[6] In the following results concerning conformal holonomy, the manifold need not be conformally spin.

(M,g) be an Einstein space of dimension $n = p+q \geq 3$ with scalar curvature $\mathrm{scal}^g \neq 0$, and let $(C(M), g_C)$ denote the metric cone $C(M) := \mathbb{R}^+ \times M$ over (M,g) with the Ricci-flat metric

$$(g_C)_{(t,x)} := \mathrm{sgn}(\mathrm{scal}^g)\left(\frac{\mathrm{scal}^g}{n(n-1)} t^2 g_x + dt^2\right). \tag{3.14}$$

We then have that

$$Hol_x(M,[g]) \cong Hol_{(1,x)}(C(M), g_C) \subset O(p+1,q+1). \tag{3.15}$$

If (M,g) is a Ricci-flat Einstein space, it holds that

$$Hol_x(M,[g]) = Hol_x(M,g) \ltimes \mathbb{R}^{n-k} \subset O(p+1,q+1), \tag{3.16}$$

where k denotes the number of linearly independent parallel vector fields on (M,g). There is the following conformal analogue of the local de Rham splitting Theorem for pseudo-Riemannian manifolds:

Theorem 3.17 ([Lei07],chapter 1;[BJ10],Thm.2.3.2) *Let (M,c) be a simply-connected conformal manifold of signature (p,q) with $n = p+q \geq 3$ and suppose that there is a $k-$dimensional, nondegenerate, $Hol_x(M,c)$ invariant subspace $V^k \subset \mathcal{T}_x(M) \cong \mathbb{R}^{p+1,q+1}$, where $2 \leq k \leq n$. Then there is an open, dense subset $\widetilde{M} \subset M$, satisfying that for all $z \in \widetilde{M}$ there is a neighbourhood $U(z)$ and a metric $g \in c_{|U(z)}$ such that $(U(z),g)$ is isometric to a product $(M_1,g_1) \times (M_2,g_2)$ of Einstein spaces of dimensions $k-1$ and $n-(k-1)$. If $k \neq 2, n$, then the scalar curvatures are related by*

$$scal^{g_1} = -\frac{(k-1)(k-2)}{(n-k+1)(n-k)} scal^{g_2} \neq 0.$$

Let (M_i,g_i) have signature (p_i,q_i) and assume that dim $M_i \geq 3$. Assume moreover wlog. that $scal^{g_1} > 0$. Then the following holds up to conjugation in $O(p+1,q+1)$:

$$Hol(U(z),c) \cong \underbrace{Hol(M_1,[g_1])}_{\subset O(p_1,q_1+1) \subset O(p_1+1,q_1+1)} \times \underbrace{Hol(M_2,[g_2])}_{\subset O(p_2+1,q_2) \subset O(p_2+1,q_2+1)}.$$

Remark 3.18 The last statements together with the algebraic fact that the only connected Lie subgroup of $SO(1,n+1)$ acting irreducible on $\mathbb{R}^{1,n+1}$ is $SO^+(1,n+1)$ are the starting points for classification of conformal holonomy groups of simply-connected conformal manifolds in the Riemannian case: If $Hol^0(M,c) \neq SO^+(1,n+1)$, there must be a $Hol(M,c)$ invariant subspace. We can then apply one of the last statements and in the end one has to classify the conformal holonomy groups of Einstein spaces (M,g) such that $[g]$ cannot be represented by a product metric. This classification is given in [Arm07].

Finally [Lei06] and [LN12a] studies the situation where the conformal holonomy representation fixes a totally lightlike subspace of dimension 2 and calls the occurring geometries conformal pure radiation metrics with parallel rays.

Theorem 3.19 *Let (M,c) be a pseudo-Riemannian conformal structure of dimension $n \geq 3$. Then $Hol_x(M,c)$ fixes a totally lightlike nullplane if and only if on an open and dense subset of M, there is a metric $g \in c$ and a null line $L \subset TM$ such that L is parallel wrt. ∇^g and $Ric^g(TM) \subset L$.*

Consequently, we see that the most involved situation when dealing with non-irreducibly acting conformal holonomy occurs when $Hol_x(M,c)$ fixes a totally lightlike subspace of dimension ≥ 3, as it may occur in the case of parallel spin tractors by (3.13) if $p \geq 2$. Up to now there is no geometric description of this situation. In view of this, the next two statements close this gap, and together with Theorems 3.16, 3.17 and 3.19 they give a complete geometric description of conformal structures admitting non-irreducibly acting conformal holonomy[7]:

Proposition 3.20 *Let (M,c) be a conformal manifold of dimension $n \geq 3$ and let $\mathcal{H} \subset \mathcal{T}(M)$ be a totally lightlike distribution of dimension $k \geq 1$ which is parallel wrt. the Cartan connection ∇^{nc}. Then there is an open, dense subset $\widetilde{M} \subset M$ such that for every point $x \in \widetilde{M}$ there is an open neighbourhood $U_x \subset \widetilde{M}$ and a metric $g \in c_{|U_x}$ such that wrt. the metric identification Φ^g (cf. (2.23)) $\mathcal{H}$ is locally given by*

$$\mathcal{H}_{|U_x} \stackrel{g}{=} span\left(\begin{pmatrix}0\\K_1\\0\end{pmatrix},...,\begin{pmatrix}0\\K_{k-1}\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

for lightlike vector fields $K_i \in \mathfrak{X}(U_x)$ which define a conformally invariant distribution $L = span\ (K_1,...,K_{k-1}) \subset TU_x$ of rank $k-1$ on U_x.

Proof. If $k = 1,2$, this statement is proved in [BJ10] and [LN12a], respectively[8]. Consequently, we may assume that $k \geq 3$. As a preparation, consider for arbitrary $g \in c$ the map $\Phi^g : \mathcal{T}(M) \to \mathcal{I}_- \oplus TM \oplus \mathcal{I}_+ = \mathcal{T}(M)_g$. We set $\mathcal{I}_- =: \mathcal{I}$ and observe from the transformation formula (2.27) that this tractor null line which defines the conformal structure does not depend on the choice of $g \in c$. In this proof, we will for fixed g always identify $\mathcal{T}(M)$ with $\mathcal{T}(M)_g$ without writing Φ^g explicitly. Moreover, we introduce the g–dependent projection $\text{pr}_{TM} : \mathcal{T}(M) \stackrel{g}{=} \mathcal{I}_- \oplus TM \oplus \mathcal{I}_+ \to TM$. Note however, that by (2.27), for every subbundle $\mathcal{V} \subset \mathcal{I}_-^{\perp} \stackrel{g}{=} \mathcal{I}_- \oplus TM$, the image $\text{pr}_{TM}(\mathcal{V}) \subset TM$ does not depend on the choice of $g \in c$.
We set $\mathcal{L} := \mathcal{I}^{\perp} \cap \mathcal{H}$, where $\perp$ is taken wrt. the standard tractor metric. By (2.24) we have that with respect to $g \in c$ it holds that $\mathcal{L} = \left\{X \in \mathcal{H} \mid X = \begin{pmatrix}\alpha\\Y\\0\end{pmatrix}\right\}$. It follows that $L := \text{pr}_{TM}\mathcal{L} \subset TM$ is conformally invariant. With these introductory remarks in mind, the proof goes as follows:

Step 1: We claim that there is an open, dense subset[9] $\widetilde{M} \subset M$ such that $\text{rk}\ \mathcal{L}_{|\widetilde{M}} = k-1$: Note that $\mathcal{L} \neq \{0\}$ as otherwise $\mathcal{H}$ would have rank 1. Assume that there is an open set U in M on which $\text{rk}\ \mathcal{L}_{|U} = k$. We fix an arbitrary metric $g \in c$. By definition of $\mathcal{L}$, we have that $\mathcal{H} \cap \mathcal{I}^{\perp} = \mathcal{L} = \mathcal{H}$ on U from which $\mathcal{H}_{|U} \subset \mathcal{I}^{\perp}_{|U} \stackrel{g}{=} (\mathcal{I}_- \oplus TM)_{|U}$ follows. Now let

[7] Note that in case of non-irreducibly acting conformal holonomy with invariant subspace $V \subset \mathbb{R}^{p+1,q+1}$ one either has that V is nondegenerate, which is covered by the previous statements, or one can pass to a totally lightlike, nontrivial subspace $\widetilde{V} := V \cap V^{\perp}$ which is also fixed by the conformal holonomy representation. This case is solved in full generality in this section.

[8] Parts of the notations in this proof follow [LN12a].

[9] In this proof, in order to keep notation short, whenever we restrict our considerations to an open, dense subset of M we again call it M.

$\underline{L} \stackrel{g}{=} \begin{pmatrix} \rho \\ Y \\ 0 \end{pmatrix} \in \Gamma(\mathcal{H}_U)$ be an arbitrary section of $\mathcal{H}$. As $\mathcal{H}$ is parallel, we must have that $\nabla^{nc}_X \underline{L} \in \Gamma(\mathcal{H}_{|U}) \subset \Gamma(\mathcal{I}^{\perp}_{|U})$ for all $X \in TU$. However, by (2.25) we get that

$$\nabla^{nc}_X \underline{L} = \begin{pmatrix} * \\ * \\ -g(X,Y) \end{pmatrix} \ \forall X \in TU,$$

which means that $Y = 0$ and $k = \text{rk } \mathcal{H} = 1$. Consequently, there is an open, dense subset (which we again call M) over which $0 < \text{rk } \mathcal{L} < k$. Now let $x \in M$ and fix a basis $L_1, ..., L_s$ of $\mathcal{L}_x$, where $s = s(x) \leq k-1$. We may add tractors $Z_l = \begin{pmatrix} a_l \\ Y_l \\ 1 \end{pmatrix} \in \mathcal{T}_x(M)$ for $1 \leq l \leq k-s$ such that $L_1, ..., L_s, Z_1, ..., Z_{k-s}$ is a basis of of $\mathcal{H}_x$. We know that $k - s \geq 1$. If $k-s > 1$ we may form new basis vectors $Z_1 + Z_2$ and $Z_1 - Z_2$. However, $Z_1 - Z_2 \in \mathcal{L}_x$. Thus, $k - s = 1$, which shows that rk $\mathcal{L}_x = k-1$.

Step 2: We claim that also $L := \text{pr}_{TM}\mathcal{L}$ has rank $k-1$ locally around each point $x \in M$. To this end, let $g \in c$ be arbitrary. Then we choose generators of $\mathcal{L}$ around x such that locally $\mathcal{L} \stackrel{g}{=} \text{span}\left(\begin{pmatrix} \sigma_1 \\ \widetilde{K}_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} \sigma_{k-1} \\ \widetilde{K}_{k-1} \\ 0 \end{pmatrix} \right)$. As $k > 2$, we may assume that $\widetilde{K}_1(x) \neq 0$. We may then at the same time also assume that $\sigma_1(x) \neq 0$. Otherwise, we find $f \in C^{\infty}(M)$ with $\widetilde{K}_1(f)(x) \neq 0$ and consider the metric $\widetilde{g} = e^{2f} g$ instead (cf. (2.27)). Moreover, we may by multiplying the generators with nonzero functions assume that there is a neighbourhood U of x on which $\sigma_1 \equiv 1$ and $|\sigma_i| < 1$ for $i = 2, ..., k-1$. By applying elementary linear algebra to the generators, we then see that there are lightlike vector fields $K_i \in \mathfrak{X}(U)$ for $i = 1, ..., k-1$ with $K_1(x) \neq 0$ such that wrt. g on U

$$\mathcal{L} \stackrel{g}{=} \text{span}\left(\begin{pmatrix} 1 \\ K_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ K_2 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ K_{k-1} \\ 0 \end{pmatrix} \right). \tag{3.17}$$

If K_1 was contained in the span of the $K_{i>1}$, we would have that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{L} \subset \mathcal{H}$. However, as by Step 1 $\mathcal{H}$ must also contain a tractor of the form $\begin{pmatrix} a \\ X \\ 1 \end{pmatrix}$ not lying in $\mathcal{L}$, this contradicts $\mathcal{H}$ being totally lightlike. Consequently, there is an open neighbourhood of x in M such that the so constructed vectors $K_1, ..., K_{k-1}$ are linearly independent and as pointwise $L = \text{span}(K_1, ..., K_{k-1})$ this shows that there is an open and dense subset of M on which the rank of L is maximal.

Step 3: It follows directly from the various definitions that

$$\text{pr}_{TM}\left(\mathcal{L}^{\perp} \cap \mathcal{I}^{\perp}\right) = L^{\perp}. \tag{3.18}$$

Moreover, note that $\mathcal{I} \subset \mathcal{L}^\perp$. By definition, $\mathcal{L} \subset \mathcal{H}$, from which $\mathcal{H}^\perp \subset \mathcal{L}^\perp$ follows. As by Step 1 $\mathcal{L} = \mathcal{H} \cap \mathcal{I}^\perp$ has codimension 1 in $\mathcal{H}$, the line $\mathcal{I}$ cannot lie in $\mathcal{H}^\perp$, i.e. $\mathcal{H}^\perp \cap \mathcal{I} = \{0\}$. A dimension count thus shows that

$$\mathcal{L}^\perp = \mathcal{H}^\perp \oplus \mathcal{I}. \tag{3.19}$$

(3.18) and (3.19) imply that

$$L^\perp = \mathrm{pr}_{TM}\left(\mathcal{H}^\perp \cap \mathcal{I}^\perp\right).$$

Step 4: In the setting of Step 2 we again fix $x \in M$, consider the local representation (3.17) of $\mathcal{L}$ wrt. some fixed g around x and set $L' := \text{span}\,(K_2, ..., K_{k-1})$[10]. Both L and L' are integrable distributions. To see this, let $i, j \in \{2, ..., k-1\}$. As $\mathcal{H}$ is parallel and totally lightlike we have that $\nabla^{nc}_{K_i}\begin{pmatrix} 0 \\ K_j \\ 0 \end{pmatrix} = \begin{pmatrix} K^g(K_i, K_j) \\ \nabla^g_{K_i} K_j \\ -g(K_i, K_j) \end{pmatrix} \in \Gamma(\mathcal{L})$. Switching the roles of i and j and taking the difference yields $\begin{pmatrix} 0 \\ [K_i, K_j] \\ 0 \end{pmatrix} \in \Gamma(\mathcal{L})$. Thus $[K_i, K_j] \in L'$. Similarly, one shows with the same argument that even

$$[K_1, L'] \subset L'. \tag{3.20}$$

In particular, L is integrable, too.

Step 5: We now apply Frobenius Theorem to $L \subset TM$: For every (fixed) point y of (an open and dense subset of) M we find a local chart $(U, \varphi = (x_1, ..., x_n))$ centered at y with $\varphi(U) = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid |x_i| < \epsilon\}$ such that the leaves $A_{c_k,...,c_n} = \{a \in U \mid x_k(a) = c_1, ..., x_n(a) = c_n\} \subset U$ are integral manifolds for L for every choice of c_j with $|c_j| < \epsilon$. It holds that $L_U = \text{span}\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{k-1}}\right)$ and moreover the coordinates may be chosen such that $K_1 = \frac{\partial}{\partial x_1}$ over U (cf. [War71]). After applying some linear algebra to the generators of L', where L' is chosen wrt. some $g \in c$ as in Step 4 and restricting U if necessary, we may assume that generators of L' are given on U by

$$K_{i \geq 2} = \alpha_i \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_i} \tag{3.21}$$

for certain smooth functions $\alpha_i \in C^\infty(U)$ for $i = 2, ..., k-1$. The integrability condition (3.20) implies that $\left[\frac{\partial}{\partial x_1}, \alpha_i \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_i}\right] = \frac{\partial \alpha_i}{\partial x_1} \cdot \frac{\partial}{\partial x_1} \in L'$, giving that

$$\frac{\partial \alpha_i}{\partial x_1} = 0 \text{ for } i = 2, ..., k-1. \tag{3.22}$$

The integrability of L' and (3.22) then yield that for $i, j = 2, ..., k-1$

$$[K_i, K_j] \overset{(3.21),(3.22)}{=} \left(\frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j}\right) \cdot \frac{\partial}{\partial x_1} \in L',$$

from which by (3.21) follows that

$$\frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j} = 0 \text{ for } i, j = 2, ..., k-1. \tag{3.23}$$

[10]Note that in contrast to L, the distribution L' depends on the choice of $g \in c$.

For fixed $c_k, ..., c_n$ as above we consider the submanifold $A_{c_k,...,c_n}$ and the differential form

$$\alpha_{c_k,...,c_n} := -\sum_{i=1}^{k-1} \alpha_i dx_i \in \Omega^1 \left(A_{c_k,...,c_n}\right), \tag{3.24}$$

where the $\alpha_{i\geq 2}$ are restrictions of the functions appearing in (3.21) to $A_{c_k,...,c_n}$ and we set $\alpha_1 \equiv -1$. (3.22) and (3.23) precisely yield that $d\alpha_{c_k,...,c_n} = 0$. Whence, there exists by the Poincaré Lemma (applied to a sufficiently small simply-connected neighbourhood) a unique $\sigma_{c_k,...,c_n} \in C^\infty\left(A_{c_k,...,c_n}\right)$ with $\sigma_{c_k,...,c_n}(\varphi^{-1}(0,...,0,c_k,...,c_n)) = 0$ and $\alpha_{c_k,...,c_n} = d\sigma_{c_k,...,c_n}$, which translates into

$$\begin{aligned} \frac{\partial \sigma_{c_k,...,c_n}}{\partial x_1} &= 1, \\ \frac{\partial \sigma_{c_k,...,c_n}}{\partial x_i} &= -\alpha_i \text{ for } i = 2, ..., k-1. \end{aligned}$$

We then define $\sigma \in C^\infty(U)$ via $\sigma(\varphi^{-1}(x_1, ..., x_n)) := \sigma_{x_k,...,x_n}(\varphi^{-1}(x_1, ..., x_n))$ and observe that on U

$$\begin{aligned} \frac{\partial \sigma}{\partial x_1} &= 1, \\ \frac{\partial \sigma}{\partial x_i} &= -\alpha_i \text{ for } i = 2, ..., k-1. \end{aligned} \tag{3.25}$$

Step 6: The construction of the generators K_i (3.21) and the properties (3.25) of σ imply that on U we have $K_1(\sigma) = 1$ and $K_i(\sigma) = 0$ for $i = 2, ..., k-1$. We now consider the rescaled metric $\widetilde{g} = e^{2\sigma} g$ on U. The transformation formula (2.27) and (3.25) then show that wrt. this metric $\mathcal{L}$ is given by

$$\mathcal{L}_U = \text{span}\left(\begin{pmatrix} 0 \\ K_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ K_{k-1} \\ 0 \end{pmatrix}\right). \tag{3.26}$$

Step 7: Let $g \in c$ be any local metric on $U \subset M$ for which (3.26) holds. We may add one generator $\begin{pmatrix} \beta \\ K \\ 1 \end{pmatrix} \in \Gamma(U, \mathcal{H})$ such that pointwise (wrt. g) $\mathcal{H} = \mathcal{L} \oplus \text{span}\begin{pmatrix} \beta \\ K \\ 1 \end{pmatrix}$. It follows that $K \in L^\perp$ as $\mathcal{H}$ is totally lightlike. By step 3 there exists a smooth function b on U with $K = \text{pr}_{TM}\begin{pmatrix} b \\ K \\ 0 \end{pmatrix}$ and $\begin{pmatrix} b \\ K \\ 0 \end{pmatrix} \in \mathcal{H}^\perp$. As $\mathcal{H}$ is lightlike, (2.24) yields that $\beta + g(K, K) = 0$ as well as $b + g(K, K) = 0$, i.e. $b = \beta$. Therefore we have that $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathcal{H}^\perp$ over U. However, this implies that $b = \beta = 0$ and we obtain

$$\mathcal{H}_U \overset{g}{=} \text{span}\left(\begin{pmatrix} 0 \\ K_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ K_{k-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ K \\ 1 \end{pmatrix}\right). \tag{3.27}$$

In the following steps we will improve the fixed metric g satisfying (3.26) within the conformal class further in such a way that K can be chosen to be zero. This goes as follows: Let $X \in \Gamma(L)$ be an arbitrary, nonzero section. We have for $Y \in TM$ that

$$\nabla^{nc}_Y \begin{pmatrix} 0 \\ X \\ 0 \end{pmatrix} \overset{(2.25)}{=} \begin{pmatrix} * \\ \nabla^g_Y X \\ -g(X,Y) \end{pmatrix} \in \Gamma(\mathcal{H}),$$

yielding $\nabla^g_T X \in \Gamma(L)$ for $T \in X^\perp$ and for perpendicular directions

$$\nabla^g_Z X = l - K \tag{3.28}$$

for some $l \in L$, where $g(X,Z) = 1$. Thus, if $g \in c$ can be chosen such that (3.26) and additionaly $\nabla^g_Y X \in \Gamma(L)$ hold for every $Y \in TM$, it holds that $K \in L$ and we can obviously rearrange the generators in (3.27) such that the Proposition follows. Steps 8 and 9 are a preparation for the construction of this desired metric.

Step 8: Wrt. $g \in c$ a metric satisfying (3.26), let $\begin{pmatrix} \rho \\ V \\ 0 \end{pmatrix} \in \Gamma(\mathcal{H}^\perp \cap \mathcal{I}^\perp)$. Further, let $X \in \Gamma(L)$ be nonzero and let Z be a vector field with $g(X,Z) = 1$. As $\mathcal{H}$ is parallel and lightlike, we have

$$0 = \langle \nabla^{nc}_Z \begin{pmatrix} 0 \\ X \\ 0 \end{pmatrix}, \begin{pmatrix} \rho \\ V \\ 0 \end{pmatrix} \rangle_{\mathcal{T}} = -\rho + g(\nabla^g_Z X, V). \tag{3.29}$$

Let $U \in L^\perp$. Further differentiation yields

$$\nabla^{nc}_U \nabla^{nc}_Z \begin{pmatrix} 0 \\ X \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ \nabla^g_U \nabla^g_Z X + K^g(X,Z) \cdot U + K^g(U)^\sharp \\ -g(U, \nabla^g_Z X) \end{pmatrix} \in \Gamma(\mathcal{H}).$$

Pairing with $\begin{pmatrix} \rho \\ V \\ 0 \end{pmatrix} \in \Gamma(\mathcal{H}^\perp \cap \mathcal{I}^\perp)$ leads to

$$0 = -\rho \cdot g(U, \nabla^g_Z X) + g(\nabla^g_U \nabla^g_Z X, V) + K^g(U,V) + K^g(X,Z) \cdot g(U,V). \tag{3.30}$$

It follows from (3.29) and (3.30) that the bilinear form

$$\Gamma(L^\perp) \times \Gamma(L^\perp) \ni (U,V) \mapsto g(\nabla^g_U \nabla^g_Z X, V) \tag{3.31}$$

is symmetric.

Step 9: Let $g \in c$ be a metric satisfying (3.26). Locally, we have that $L = \operatorname{span}(K_1, ..., K_{k-1})$ and $L^\perp = \operatorname{span}(K_1, ..., K_{k-1}, E_1, ..., E_l)$, where $l = n - 2k + 2$ and the E_i are vector fields on $U \subset M$ which are orthogonal to the K_i and satisfy $g(E_i, E_j) = \pm 1 \cdot \delta_{ij}$. $L^\perp$ is an integrable distribution: By Step 3 there exist functions ρ_j such that E_j is the projection of $\begin{pmatrix} \rho_j \\ E_j \\ 0 \end{pmatrix} \in \Gamma(\mathcal{H}^\perp \cap \mathcal{I}^\perp)$ to TM for $j = 1, ..., n - 2k + 2$. As also $\mathcal{H}^\perp$ is parallel, (2.25) yields for $i \neq j$ that

$$\nabla^{nc}_{E_i} \begin{pmatrix} \rho_j \\ E_j \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ \nabla^g_{E_i} E_j + \rho_j E_i \\ 0 \end{pmatrix} \in \Gamma\left(\mathcal{H}^\perp \cap \mathcal{I}^\perp\right).$$

It follows that $\nabla^g_{E_i}E_j \in L^\perp$. With the same argumentation, one finds that also $\nabla^g_{E_i}K_j, \nabla^g_{K_j}E_i \in L^\perp$ for $i = 1,...,n-k+1$, $j = 1,...k-1$. This yields together with integrability of L and torsion-freeness of ∇^g the integrability of $L^\perp$.

Step 10: As $L \subset L^\perp$ and both are integrable distributions, we can by Frobenius Theorem (cf. Step 5) applied first to $L^\perp$ and then to each leaf of $L^\perp$ find around every point local coordinates

$$(U, \varphi = (x_1,...,x_{k-1},y_1,...,y_{n-2k+2},z_1,...,z_{k-1}))$$

such that $(x_1,...,x_{k-1})$ parametrizes integral manifolds for L and $(x_1,...,x_{k-1},y_1,...,y_{n-2k+2})$ parametrizes integral manifolds for $L^\perp$.
Let $\sigma \in C^\infty(U)$ be an arbitrary function depending on $(y_1,...,y_{n-2k+2},z_1,...,z_{k-1})$ only and set $\widetilde{g} = e^{2\sigma}g$. Clearly, $X(\sigma) = 0$ for every $X \in L = \text{span}\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_{k-1}}\right)$ and thus the tractor $\begin{pmatrix} 0 \\ X \\ 0 \end{pmatrix} \in \Gamma(\mathcal{L})$ is wrt. $\widetilde{g}$ given by (cf.(2.27)) $\begin{pmatrix} 0 \\ \widetilde{X} \\ 0 \end{pmatrix} \in \Gamma(\mathcal{L})$ for some $\widetilde{X} \in L$. This means that also $\widetilde{g}$ satisfies (3.26).
We set $X := \frac{\partial}{\partial x_1} \in \Gamma(L)$ and fix a vector field Z such that $g(X,Z) = 1, g(\frac{\partial}{\partial x_{i>1}}, Z) = g(\frac{\partial}{\partial y_j}, Z) = 0$. We want to show that

$$g(\nabla^{\widetilde{g}}_Z X, Y) = 0 \text{ for every } Y \in L^\perp, \tag{3.32}$$

from which $\nabla^{\widetilde{g}}_Z X \in \Gamma(L)$ follows. To this end, we calculate with the well-known transformation formula $\nabla^{\widetilde{g}}_B A = \nabla^g_B A + d\sigma(B)A + d\sigma(A)B - g(A,B)\cdot \text{grad}^g\sigma$ for the Levi-Civita connection that

$$\begin{aligned} g(\nabla^{\widetilde{g}}_Z X, Y) &= g(\nabla^g_Z X, Y) + d\sigma(Z)\underbrace{g(X,Y)}_{=0} + \underbrace{d\sigma(X)}_{=0} g(Z,Y) - g(\text{grad}^g\sigma, Y) \\ &= \left(g(\nabla^g_Z X, \cdot) - d\sigma\right)(Y), \end{aligned} \tag{3.33}$$

where $Y \in L^\perp$. On the other hand, we calculate for $U, V \in \Gamma(L^\perp)$

$$\begin{aligned} d(g(\nabla^g_Z X, \cdot))(U,V) &= U(g(\nabla^g_Z X, V)) - V(g(\nabla^g_Z X, U)) - g(\nabla^g_Z X, [U,V]) \\ &= g(\nabla^g_U \nabla^g_Z X, V) - g(\nabla^g_V \nabla^g_Z X, U) \overset{(3.31)}{=} 0. \end{aligned} \tag{3.34}$$

To evaluate this further, we introduce $\theta := g(\nabla^g_Z X, \cdot) \in \Omega^1(U)$. As moreover $g(\nabla^g_Z X, l) = 0$ for every $l \in L$ (cf. (3.28)), there exist local functions $\alpha_i, \beta_j \in C^\infty(U)$ such that $\theta = \sum_i \alpha_i dy_i + \sum_j \beta_j dz_j$. Let us define $\widetilde{\theta} := \sum_i \alpha_i dy_i$ and let $\widetilde{\theta}_{A_{c_1,...,c_{k-1}}}$ denote its restriction to the leaf $A_{c_1,...,c_{k-1}} := \{\varphi(x_1,...,x_{k-1},y_1,...,y_{n-2k+2},c_1,...,c_{k-1}) \mid c_i = \text{const.}\}$ of $L^\perp$. Obviously, (3.34) is equivalent to $d\left(\widetilde{\theta}_{A_{c_1,...,c_{k-1}}}\right) = 0$ for all c_i. Thus, by applying the Poincaré Lemma again on a sufficiently small neighbourhood, we conclude that there are unique $\gamma_{c_1,...,c_{k-1}} \in C^\infty(A_{c_1,...,c_{k-1}})$ such that $\gamma_{c_1,...,c_{k-1}}(\varphi^{-1}(0,...,0,c_1,...,c_{k-1})) = 0$ and $d\gamma_{c_1,...,c_{k-1}} = \widetilde{\theta}_{A_{c_1,...,c_{k-1}}}$. We now specify $\sigma \in C^\infty(U)$ by setting

$$\sigma(\varphi^{-1}(x_1,....,y_{n-2k+2},z_1,...,z_{k-1})) := \gamma_{z_1,...,z_{k-1}}(\varphi^{-1}(x_1,....,y_{n-2k+2},z_1,...,z_{k-1})).$$

This construction yields for $Y \in L^\perp$

$$d\sigma(Y) = \widetilde{\theta}(Y) = \theta(Y) = g(\nabla^g_Z X, Y).$$

Letting $Y = \frac{\partial}{\partial x_i}$ and using $\nabla^g_Z X \in \Gamma(L^\perp)$, cf. (3.28), yields $\frac{\partial \sigma}{\partial x_i} = 0$, i.e. σ does not depend on $(x_1, ..., x_{k-1})$. Consequently, we get from (3.33) for this choice of σ that (3.32) holds. However, as remarked at the end of Step 7, this already proves the Proposition. □

We study some consequences. In the setting of Proposition 3.20 we have that $\mathcal{H}$ is parallel iff $\mathcal{H}^\perp$ is parallel. Locally, we have wrt. the metric g and the distribution L appearing in Proposition 3.20 that $\mathcal{H}^\perp = \text{span}\left(\begin{pmatrix} 0 \\ X \\ \tau \end{pmatrix} \mid X \in L^\perp \right)$. It follows that $\mathcal{H}^\perp$ is parallel iff

$$\nabla^{nc}_Y \begin{pmatrix} 0 \\ X \\ \tau \end{pmatrix} = \begin{pmatrix} K^g(X,Y) \\ \nabla^g_Y X - \tau K^g(Y) \\ Y(\tau) - g(X,Y) \end{pmatrix} \in \Gamma(U, \mathcal{H}^\perp)$$

for all $X \in \Gamma(U, L^\perp)$ and $Y \in \mathfrak{X}(U)$. Clearly, this is equivalent to parallelism of L and $K^g(X,Y) = 0$ for all $X \in L^\perp$, i.e. $K^g(TU) \subset L$. Together with the next Lemma, these two conditions are equivalent to parallelism of L and $Ric^g(TU) \subset L$.

Lemma 3.21 *Assume that for a pseudo-Riemannian manifold (M,g) one has a non-trivial totally lightlike $(k-1)$–dimensional distribution $L \subset TM$ for which $K^g(TM) \subset L$. Then $scal^g = 0$.*

Proof. For fixed $x \in M$ we introduce a basis $(X_1, ..., X_{k-1}, X'_1, ..., X'_{k-1}, E_1, ..., E_l)$ of T_xM, where $L_x = \text{span}\{X_1, ..., X_{k-1}\}$, $g(X_i, X'_j) = \delta_{ij}, g(X'_i, X'_j) = 0$, $g(E_i, E_j) = \epsilon_i \delta_{ij}$ and $g(E_i, X^{(\prime)}_j) = 0$. It follows that

$$\text{scal}^g(x) = 2 \sum_{j=1}^{k-1} Ric^g(X_j, X'_j) + \sum_{i=1}^{l} \epsilon_i Ric^g(E_i, E_i). \tag{3.35}$$

By definition of the Schouten tensor, we have that $Ric^g = \frac{1}{2(n-1)} \cdot \text{scal}^g - (n-2) \cdot K^g$. Inserting this into (3.35) yields

$$\begin{aligned} \text{scal}^g(x) =& \frac{1}{n-1}(k-1) \cdot \text{scal}^g(x) - 2(n-2) \cdot \sum_{j=1}^{k-1} \underbrace{K^g(X_j, X'_j)}_{=0} \\ &+ \frac{1}{2(n-1)} \cdot (n - 2(k-1)) \cdot \text{scal}^g(x) - (n-2) \sum_{j=1}^{l} \epsilon_i \underbrace{K^g(E_i, E_i)}_{=0} \\ =& \frac{n}{2(n-1)} \cdot \text{scal}^g(x), \end{aligned}$$

i.e. $\text{scal}^g(x) = 0$. □

Finally, we have seen in the proof of Proposition 3.20 that $L = \text{pr}_{TM}\mathcal{L} \subset TM$ is a well-defined distribution of constant rank on $\widetilde{M} \subset M$ open and dense. As L is on $\widetilde{M}$ *locally* parallel wrt. certain metrics in the conformal class, this implies by the torsion-freeness of ∇^g as a *global* consequence that L is integrable on $\widetilde{M}$. Thus, altogether we have proved the following:

Theorem 3.22 *If on a conformal manifold (M,c) there exists a totally lightlike, k-dimensional parallel distribution $\mathcal{H} \subset \mathcal{T}(M)$, then there is an open and dense subset $\widetilde{M}$ of M on which the totally lightlike distribution $L := pr_{TM}(\mathcal{H} \cap \mathcal{I}_-^{\perp}) \subset TM$ is of constant rank $k-1$ and integrable. Every point $x \in \widetilde{M}$ admits a neighbourhood $U = U_x$ and a metric $g \in c_U$ such that on U:*

$$\begin{aligned} &Ric^g(TU) \subset L, \\ &L \text{ is parallel wrt. } \nabla^g, \text{ i.e. } Hol_x(U,g)L_x \subset L_x. \end{aligned} \tag{3.36}$$

Conversely, let (U,c) be a conformal manifold. Suppose that there is $g \in c$ and a $(k-1)$-dimensional totally lightlike distribution $L \subset TU$ such that (3.36) holds. Then L gives rise to a $k-$dimensional totally lightlike, parallel distribution $\mathcal{H} \overset{\Phi^g}{=} \begin{pmatrix} 0 \\ L \\ 0 \end{pmatrix} \oplus span \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathcal{T}(U)$.

Thus, one has a totally lightlike, parallel distribution in the standard tractor bundle if and only if one has locally a totally lightlike and parallel distribution of one dimension less in the tangent bundle with respect to some metric in the conformal class which additionally satisfies the curvature condition (3.36). Up to now there is no complete classification of metric holonomy groups satisfying (3.36).

Remark 3.23 If in the setting of Theorem 3.22 on (M,c) a totally lightlike, parallel subbundle $\mathcal{H} \subset \mathcal{T}(M)$ with associated totally lightlike distribution $L := \text{pr}_{TM}(\mathcal{H} \cap \mathcal{I}_-^{\perp}) \subset TM$ of constant rank on an open and dense subset $\widetilde{M}$ of M is given, one proves precisely as in [LN12a], Remark 2, that one obtains the conformally invariant integrability condition

$$W^g(L, L^{\perp}, \cdot, \cdot) = 0$$

for the Weyl tensor for arbitrary metric $g \in c_{\widetilde{M}}$.

Remark 3.24 Theorem 3.22 provides a natural generalization of the third case from Theorem 3.16 ($L = 0$ in this case which yields Ricci-flatness) and Theorem 3.19, where dim $L = 1$. Moreover, Theorem 3.22 also naturally generalizes results from [Lei05] where the statement is proved under the additional condition that the totally lightlike distribution $\mathcal{H} \subset \mathcal{T}(M)$ arises from a decomposable, totally lightlike twistor $k-$form[11]. However, as elaborated on in [LN12a], in general not every holonomy-invariant totally lightlike $k-$dimensional subspace gives rise to a holonomy-invariant totally lightlike $k-$form. Thus we get the same geometric structures as discussed in [Lei05] in the presence of totally lightlike twistor forms but under weaker assumptions.

Remark 3.25 It is a common feature of all statements about reducible conformal holonomy that one always has to leave out a certain set of singular points, i.e. restrict to some open and dense subset $\widetilde{M} \subset M$, as was also necessary in the proof of Proposition 3.22. The deeper reason for this has recently been discovered in [CGH14], and it is closely related to so called curved orbit decompositions of arbitrary Cartan geometries. At least in the case of a holonomy-invariant line which is the situation in Theorem 3.16, the reference can with this method also describe the geometry of the singular set $M \backslash \widetilde{M}$.

[11] By this we mean that there is a holonomy-invariant form $l_1 \wedge ... \wedge l_k$ where the l_i span a totally lightlike $k-$dimensional subspace in $\mathbb{R}^{p+1,q+1}$. Clearly, this space is then also holonomy-invariant

Let us study the local geometries occurring in Theorem 3.22 in more detail: Pseudo-Riemannian geometries admitting parallel, totally lightlike distributions are called Walker manifolds and have been studied in [VRG09], for instance. As the geometries from Theorem 3.22 will occur quite frequently in the classification procedure for twistor spinors, we shall give them a special name.

Definition 3.26 *A pseudo-Riemannian manifold (M,g) admitting a parallel, totally lightlike distribution $L \subset TM$ of rank r, satisfying additionally that $Ric^g(TM) \subset L$ is called Ricci-isotropic pseudo-r-Walker manifold.*

In general, for every n–dimensional Walker manifold (M,g) with parallel, r–dimensional, totally lightlike distribution $L \subset TM$, there are locally around each point coordinates $(x_1,...,x_n)$ such that wrt. the basis $\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right)$ the metric tensor reads (cf. [VRG09])

$$g_{ij} = \begin{pmatrix} 0 & 0 & Id_r \\ 0 & A & H \\ Id_r & H^T & B \end{pmatrix},$$

where A is a symmetric $(n-2r)\times(n-2r)$ matrix, B is a symmetric $r \times r$ matrix and H is a $(n-2r)\times r$ matrix. Moreover, A and H do not depend on $(x_1,...,x_r)$, and in these coordinates, L is given by

$$L = \text{span}\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_r}\right).$$

Example 3.27 Let φ be a parallel spinor on a pseudo-Riemannian spin manifold (M,g). Then $L := \ker \varphi = \{X \in TM \mid X\cdot\varphi = 0\} \subset TM$ is totally lightlike and parallel. (3.1) translates into $Ric^g(TM) \subset L$. For small dimensions all Ricci-isotropic pseudo-r-Walker metrics arising in this way have been classified in [Bry00]. The orbit structure of $\Delta_{p,q}$ encodes which values for r =dim L are possible in these cases.

3.4 A holonomy-description of twistor spinors equivalent to parallel spinors

The previous results regarding conformal holonomy naturally apply to the case of twistor spinors on conformal spin manifolds. Let $\psi \in \Gamma(\mathcal{S}(M))$ be a parallel spin tractor on $(M^{p,q},c)$ and for $g \in c$ let $\varphi := \widetilde{\Phi}^g(proj^g_+\psi) \in \Gamma(S^g)$ be the associated twistor spinor. We consider the totally lightlike and ∇^{nc}–parallel distribution ker $\psi \subset \mathcal{T}(M)$ as introduced in (3.13). In complete analogy, if even φ is parallel wrt. some $g \in c$, we get a totally lightlike, parallel distribution ker $\varphi \subset TM$. One then has the following important consequence from Proposition 3.20:

Proposition 3.28 *If $\psi \in \Gamma(\mathcal{S}(M))$ is a parallel spin tractor with ker $\psi \neq \{0\}$, then there is an open and dense subset $\widetilde{M} \subset M$ such that on $\widetilde{M}$ the associated twistor spinor $\varphi := \widetilde{\Phi}^g\,(proj^g_+\psi)$ is locally conformally equivalent to a parallel spinor. Conversely, if $\varphi \in \Gamma(S^g)$ is a parallel spinor for some $g \in c$, then $\psi := \left(\widetilde{\Phi}^g \circ proj^g_+\right)^{-1}\varphi \in Par(\mathcal{S}(M),\nabla^{nc})$ satisfies ker $\psi \neq \{0\}$. In both cases, it holds that the constant(!) dimensions of the* *distributions ker* $\psi \subset \mathcal{T}(\widetilde{M})$ *and ker* $\varphi \subset T\widetilde{M}$ *are on* $\widetilde{M}$ *related by*

$$dim\ ker\ \psi = dim\ ker\ \varphi + 1. \tag{3.37}$$

Proof. We notice that Proposition 3.20 applied to $\mathcal{H}$ =ker ψ yields the desired $\widetilde{M}$ and for $x \in \widetilde{M}$ a neighbourhood U and a local metric $g = g_U \in c_U$ such that wrt. g we have $s_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \ker \psi_{|U}$. If we decompose ψ on U wrt. the metric g as in Theorem 3.3, i.e. $\psi_{|U} = [[\widetilde{\sigma}^g(\widetilde{u}), e], e_- \cdot w + e_+ \cdot w]$ for some function $w : U \to \Delta_{p+1,q+1}$ and a local section $\widetilde{u} : U \to \mathcal{Q}^g_+$, the condition $s_+ \cdot \psi = 0$ yields that $e_+ \cdot e_- \cdot w = 0$ on U which by multiplication with e_- implies that $e_- \cdot w = 0$. However, by Theorem 3.3 it follows that on U we have $D^g\varphi = -n \cdot \widetilde{\Phi}^g(proj^g_-(\psi)) = 0$. Thus, φ is on U both harmonic and a twistor spinor and therefore parallel wrt. g. Conversely, by the same argumentation every parallel spinor $\varphi \in \Gamma(S^g)$ satisfies $s_+ \in \ker \psi$. (3.37) follows directly from the local metric description of ker ψ from Proposition 3.22: Let $K \in TM \subset \mathcal{T}(M)_g$ with $K \cdot \psi = 0$, then $K \cdot s_\pm = -s_\pm \cdot K$ implies that already $K \cdot \widetilde{\Phi}^g(proj^g_\pm \psi) = 0$, i.e. $K \in \ker \varphi$. Thus, one has in the language of Proposition 3.22 that $L = \mathrm{pr}_{TM}(\ker \psi \cap \mathcal{I}^\perp_-) = \ker \varphi$ from which (3.37) follows. □

Remark 3.29 Proposition 3.28 yields (locally) an *equivalent* characterization of those twistor spinors which are (locally) conformally equivalent to parallel spinors in terms of conformally invariant objects, i.e. on a local level one has the following correspondences:

$$\begin{array}{ccc} \boxed{\varphi \in \ker P^g} & \Leftrightarrow & \boxed{v \in \Delta_{p+1,q+1} \text{ with } \lambda_*^{-1}(\mathfrak{hol}(M,[g])) \cdot v = 0} \\ & \Downarrow \textit{reduces to} & \\ \boxed{\varphi \in \ker P^g \text{ parallel}} & \Leftrightarrow & \boxed{v \in \Delta_{p+1,q+1} \text{ with } \lambda_*^{-1}(\mathfrak{hol}(M,[g])) \cdot v = 0, \ker v \neq \{0\}} \end{array}$$

In particular, this shows that $Hol(M,c)$ never acts irreducibly if there is a metric with parallel spinor in the conformal class.

Remark 3.30 In terms of the original data, i.e. without using tractor notation, Proposition 3.28 can be rephrased as follows: Note that wrt. the decomposition (2.33) of $\mathcal{S}(M) \cong S^g(M) \oplus S^g(M)$, the requirement ker $\psi(x) \neq \{0\}$ is equivalent to say that there is $x \in M$, $g \in c$ and a nontrivial triple $(\alpha, X, \beta) \in \mathbb{R} \oplus T_xM \oplus \mathbb{R} = \mathcal{T}_x(M)_g$ such that

$$\begin{aligned} X \cdot \varphi(x) + \alpha \cdot D^g\varphi(x) &= 0, \\ X \cdot D^g\varphi(x) + \beta \cdot \varphi(x) &= 0. \end{aligned}$$

This description allows several important consequences. If, for example, $D^g\varphi$ vanishes at *some* point for some metric in the conformal class, then the twistor spinor is already locally equivalent to a parallel spinor locally around every(!) point (up to a singular set).

Proposition 3.28 admits several further consequences and applications which contribute to the classification problem for local geometries admitting twistor spinors on pseudo-Riemannian manifolds. We first describe how it is related to and generalizes previous results obtained for the Riemannian and Lorentzian case:

Remark 3.31 For a Riemannian spin manifold (M^n,g) with twistor spinor φ one has that $M\backslash Z_\varphi = \{x \in M \mid \varphi(x) \neq 0\}$ is dense in M and $\left(M\backslash Z_\varphi, \widetilde{g} = \frac{1}{\|\varphi\|^4}\right)$ is an Einstein space of nonnegative scalar curvature $\widetilde{R}$. If $\widetilde{R} > 0$, then the rescaled spinor decomposes into a sum of two Killing spinors whereas in case $\widetilde{R} = 0$ one has a Ricci-flat metric with parallel spinor. Proposition 3.28 precisely describes this last Ricci-flat case in which dim ker $\psi = 1$ is maximal (cf. also Theorem 3.16).

For the Lorentzian case, Lemma 1.24 yields a relation between Proposition 3.28 and the classification results for twistor spinors on Lorentzian manifolds which were obtained using the nc-Killing form theory in [Lei07]. To this end, note that given a parallel spin tractor $\psi \in \Gamma(\mathcal{S}(M))$, the parallel tractor 2-form $\alpha^2_\psi \in \Omega^2_{\mathcal{T}}(M)$ must be (pointwise) of one of the generic types from Remark 1.25[12] where we consider the last two types in common and call them Kähler type.

Theorem 3.32 ([Lei07]; Thm.10) *Let $\varphi = \widetilde{\Phi}^g(proj^g_+\psi) \in \Gamma(S^g_{\mathbb{C}})$ be a complex twistor spinor on a Lorentzian spin manifold $(M^{1,n-1},g)$ of dimension $n \geq 3$. Then one of the following holds on an open and dense subset $\widetilde{M} \subset M$:*

1. *$\alpha^2_\psi = l_1^\flat \wedge l_2^\flat$ for standard tractors l_1, l_2 which span a totally lightlike plane.*
   In this case, φ is locally conformally equivalent to a parallel spinor with lightlike Dirac current V_φ on a Brinkmann space.

2. *$\alpha^2_\psi = l^\flat \wedge t^\flat$ where l is a lightlike, t is an orthogonal, timelike standard tractor.*
   (M,g) is locally conformally equivalent to $(\mathbb{R}, -dt^2) \times (N_1,h_1) \times \cdots \times (N_r,h_r)$, where the (N_i,h_i) are Ricci-flat Kähler, hyper-Kähler, G_2-or $Spin(7)$-manifolds.

3. *α^2_ψ is of Kähler-type at every point (cf. Remark 1.25).*
   The following cases can occur:
   a) *The dimension n is odd and the space is locally equivalent to a Lorentzian Einstein-Sasaki manifold on which the spinor is a sum of Killing spinors.*
   b) *n is even and (M,g) is locally conformally equivalent to a Fefferman space.*
   c) *There exists locally a product metric $g_1 \times g_2 \in [g]$ on M, where g_1 is a Lorentzian Einstein-Sasaki metric on a space M_1 of dimension $n_1 = 2 \cdot rk(\alpha_1(\varphi)) + 1 \geq 3$ admitting a Killing spinor and g_2 is a Riemannian Einstein metric with Killing spinor on a space M_2 of positive scalar curvature $scal^{g_2} = \frac{(n-n_1)(n-n_1-1)}{n_1(n_1-1)} scal^{g_1}$.*

Applying Lemma 1.24 to α^2_ψ reveals that ker $\psi \neq \{0\}$ occurs exactly in the first two cases of Theorem 3.32 in which we get a parallel spinor by our Proposition 3.28 as also follows from the preceding Theorem. In the third case of Theorem 3.32, it holds by Lemma 1.24 that dim ker $\psi = \{0\}$ and thus by Proposition 3.28 the spinor cannot be rescaled to a parallel spinor. In particular, combining Theorem 3.32 and Proposition 3.28 yields:

Proposition 3.33 *Twistor spinors in even dimension which define a special unitary conformal holonomy reduction, i.e. a Fefferman metric in the conformal class, or Killing*

[12] Note that as α^2_ψ is parallel, its $SO^+(p+1,q+1)$-orbit type is constant over M.

spinors in odd dimension which define a Lorentzian Einstein Sasaki structure are never locally conformally equivalent to parallel spinors.

In summary, Proposition 3.28 is in accordance with previous classification results in the Riemannian and Lorentzian case.

We describe further geometric consequences implied by Proposition 3.28. In the notation and under the assumptions of this Proposition, the fact that φ can locally be rescaled to a parallel spinor and the vanishing of the torsion of ∇^g for every $g \in c$ imply as a *global* consequence that ker $\varphi \subset TM$ is an *integrable* distribution on $\widetilde{M}$. Now fix $x \in \widetilde{M}$ and let $U \subset \widetilde{M}$ be an open neighbourhood with metric $g \in c_{|U}$ such that φ is parallel wrt. g on U. In case that $k := \text{dim ker } \varphi_{|U} > 0$, $Hol(U,g)$ acts reducible with a fixed totally lightlike $k-$dimensional subspace. Let us assume that $p \leq q$. If $k = p$, i.e. ker φ is of maximal dimension on U, it follows from Lemma 1.24 that even a totally isotropic $k-$form is fixed by the holonomy representation. If $k = p - 1$, $Hol(U,g)$ fixes a $p-$form of type $\alpha^p_\varphi = l_1^\flat \wedge ... \wedge l_{p-1}^\flat \wedge t^\flat$, where t is not lightlike and orthogonal to the l_i. As $Hol(U,g)$ acts by orthogonal transformations, it follows that even the totally lightlike form $l_1^\flat \wedge ... \wedge l_{p-1}^\flat$ is fixed by the holonomy representation. If $k = 0$ it follows from Proposition 3.28 or $Ric^g(TU) \cdot \varphi = 0$ that g is a Ricci-flat metric on U. There is a complete list of possible irreducible, non locally symmetric holonomy groups for this case as to be found in Theorem 3.2. We summarize these results as follows:

Proposition 3.34 *Let ψ be a parallel spin tractor with ker $\psi \neq \{0\}$ and for $g \in c$ let φ be the associated twistor spinor. Then there is an open, dense subset $\widetilde{M} \subset M$ such that ker φ is an integrable distribution of dimension dim ker ψ-1 on $\widetilde{M}$. Moreover, any $x \in \widetilde{M}$ admits an open neighbourhood $U \subset \widetilde{M}$ and a metric $g \in c_{|U}$ such that φ is a parallel spinor on (U,g) and one has precisely one of the following local geometries:*

1. *$k :=$ dim ker $\varphi \neq 0$. In this case, $Hol(U,g)$ acts reducible with fixed $k-$dimensional totally lightlike subspace L and $Ric^g(TU) \subset L$. Moreover, if $k = p, p-1$ there is a totally isotropic parallel $k-$form.*

2. *$k :=$ dim ker $\varphi = 0$. The space (U,g) is Ricci-flat. If it is not locally symmetric and $Hol(U,g)$ acts irreducible, then it is one of the list in Theorem 3.2.*

In this sense, the conformally invariant integer dim ker ψ encodes, if >0, the possible local geometries with parallel spinor off the open, dense subset $\widetilde{M}$.

Remark 3.35 We can view the subcase in Proposition 3.34 in which one has a totally lightlike, parallel $p-$form as a higher-signature analogue of parallel spinors on Lorentzian manifolds with lightlike, parallel Dirac currents which yield Brinkmann spaces as local geometries. We further remark that similar integrability conditions to those in Proposition 3.34 for *pure* twistor spinors have been derived independently in [TC13a] and [TC13b].

For our last application of Proposition 3.28 we consider split signatures $(m+1,m)$ or (m,m) where $\Delta^{\mathbb{C}}_{m+1,m}$ admits a real structure and we can thus restrict ourselves to real spinor fields. Geometries admitting real, parallel and (pointwise) pure spinor fields in these signatures have been classified:

Theorem 3.36 ([Kat99], Thm. 8.2) *Let (M,h) be a pseudo-Riemannian spin manifold of split signature $(m+1,m)$ admitting a real pure parallel spinor field in $\Gamma(M,S^h_{\mathbb{R}})$. Then one can find for every point in M local coordinates (x,y,z) , $x=(x_1,...,x_m)$, $y=(y^1,...,y^m)$ around this point such that*

$$h=-dz^2-4\sum_{i=1}^m dx_i dy^i-4\sum_{i,j=1}^m g_{ij}dy^i dy^j, \tag{3.38}$$

where g_{ij} are functions depending on x,y and z and satisfying

$$g_{ij}=g_{ji} \text{ for } i,j=1,...,m,\ \sum_{i=1}^m \frac{\partial g_{ik}}{\partial x_i}=0 \text{ for } k=1,...,m. \tag{3.39}$$

Conversely, if one uses (3.38) and (3.39) to define a metric h on a connected open set $U\subset\mathbb{R}^{2m+1}$, then (U,h) is spin and admits a real pure parallel spinor. $Hol(U,h)$ is contained in the image under the double covering λ of the identity component of the stabilizer of a real pure spinor which is $R^+(m+1,m)=SL(m)\ltimes N$ as given Lemma 1.21. h is not necessarily Ricci-flat. Similar statements hold in case $(p,q)=(m,m)$, where one has to omit the last coordinate etc.

At the beginning of this chapter, we have studied conformal holonomy under the additional condition that the holonomy group acts irreducibly on $\mathbb{R}^{p+1,q+1}$ and transitively on $Q^{p,q}$ and classified geometries admitting twistor spinors. We now turn to the other extremal situation: We consider split-signature conformal holonomy groups admitting a totally lightlike invariant subspace of maximal rank, derive an equivalent characterization for the existence of twistor spinors in terms of curvature quantities and completely describe the occurring geometries: The following Theorem *reverses* the mapping

$$\boxed{\psi\in\Gamma(M,\mathcal{S})\text{ parallel}}\Rightarrow\boxed{\mathcal{H}=\ker\,\psi\subset\mathcal{T}(M)\text{ lightlike and parallel}}$$

for a special case.

Theorem 3.37 *Let $(M^{m,m},c)$ be a simply-connected split-signature conformal spin manifold and let $\mathcal{H}\subset\mathcal{T}(M)$ be a $(m+1)-$dimensional totally lightlike distribution . Let $L:=pr_{TM}(\mathcal{H}\cap\mathcal{I}_-^{\perp})\subset TM$ denote the conformally invariant projection, let $g\in c$ and suppose that* [13]

1. *$\mathcal{H}$ is ∇^{nc}-parallel,*
2. *$tr\,W^g(X,Y)_{|L}=0$ for all $X,Y\in\mathfrak{X}(M)$.*

Then (M,c) admits on an open, dense subset $\widetilde{M}\subset M$ a parallel, real spin tractor $\psi\in\Gamma(\widetilde{M},\mathcal{S}_{\mathbb{R}}(\widetilde{M}))$ with $\mathcal{H}=ker\,\psi$ which is uniquely determined up to multiplication with a constant. Moreover, there is an open, dense subset $\widetilde{M}\subset M$ on which the conformal class is locally represented by a metric g as in Theorem 3.36, and it holds locally wrt. this g that $\varphi=\widetilde{\Phi}^g(proj^g_+\psi)$ is a parallel, real pure spinor on (U,g).
Conversely, if $(M^{m,m},c)$ admits a parallel spin tractor ψ such that dim $ker\,\psi=m+1$, then $\mathcal{H}:=ker\,\psi$ satisfies the conditions 1. and 2.

[13] Clearly, by the conformal transformation behaviour of the Weyl tensor, the following curvature property does not depend on the choice of $g\in c$

Remark 3.38 The above statement can be viewed as a conformal analogue for real pure parallel spinor fields on pseudo-Riemannian manifolds in [Kat99] where a totally lightlike ∇^g-parallel distribution in TM which satisfies an additional trace condition for R^g yields parallel spinors. Also the first part of the proof runs through the same lines. Moreover, we remark that our proof can be easily carried over to an analogous statement for split signatures $(m+1,m)$.

Proof. We first prove that the conditions 1. and 2. on (M,c) and $\mathcal{H}$ are equivalent to the existence of a parallel pure spin tractor: By applying Proposition 1.19 pointwise, we see that $\mathcal{S}^{\mathcal{H}} := \{\psi \in \mathcal{S}(M) \mid \mathcal{H}\cdot\psi = 0\} \subset \mathcal{S}^{\pm}_{\mathbb{R}}(M)$ is a real, 1-dimensional bundle over M. As $\mathcal{H}$ is parallel, the Cartan connection ∇^{nc} restricts to a covariant derivative ∇^{nc} on $\mathcal{S}^{\mathcal{H}}$. Clearly, a globally defined parallel section in $\mathcal{S}^{\mathcal{H}}$ would give the desired parallel spin tractor and yield that for the curvature we have $R^{\nabla^{nc},\mathcal{S}^{\mathcal{H}}} = 0$. Conversely, as M is simply-connected, the vanishing of the curvature of this line bundle would also give a globally defined parallel section. Thus , assuming 2., it suffices to show that

$$R^{\nabla^{nc},\mathcal{S}^{\mathcal{H}}}(X,Y) = 0 \Leftrightarrow \mathrm{tr}\ W^g(X,Y)_{|L} = 0. \tag{3.40}$$

We assume that $\mathcal{S}^{\mathcal{H}} \subset \mathcal{S}^{+}_{\mathbb{R}}(M)$, the proof for $\mathcal{S}^{\mathcal{H}} \subset \mathcal{S}^{-}_{\mathbb{R}}(M)$ is similar. We choose the representation of $Cl(m+1,m+1)$ from Remark 1.5 with $\epsilon_{2j-1} = -1$ and $\epsilon_{2j} = 1$[14] and consider the real, pure spinor $v := \begin{pmatrix}1\\0\end{pmatrix} \otimes ... \begin{pmatrix}1\\0\end{pmatrix} = u(1,...,1) \in \Delta^{\mathbb{R},+}_{m+1,m+1}$. By Proposition 6.20 and as pure spinors constitute a single half-spinor orbit, we can find around each $x \in M$ a local section $\widetilde{u} : U \to \overline{\mathcal{Q}}^1_+$ such that $\mathcal{S}^{\mathcal{H}}(x) = \mathbb{R}\cdot[\widetilde{u},v]$. Consider the local section $\Theta := [\widetilde{u},v]$ of $\mathcal{S}^{\mathcal{H}}$. Moreover, let $(s_1,...,s_{n+2})$ denote the local pseudo-orthonormal frame in $\mathcal{T}(M)$ defined by $s_i = \left[\overline{f}^1(\widetilde{u}), e_i\right]$ for $i = 1,...,n+2$. Note that $\mathcal{H}$ is spanned over U by $s_{2j-1}+s_{2j}$ as these tractors annihilate Θ. Then the curvature formula from Proposition 3.41 which is proved below yields that for $X,Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned} R^{\nabla^{nc},\mathcal{S}}(X,Y)\Theta &= \frac{1}{2}\sum_{i<j}\epsilon_i\epsilon_j\langle R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)s_i,s_j\rangle_{\mathcal{T}} s_i\cdot s_j\cdot\Theta \\ &= \frac{1}{2}\sum_{i<j}\epsilon_i\epsilon_j\langle R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)s_i,s_j\rangle_{\mathcal{T}}\left[\widetilde{u}, e_i\cdot e_j\cdot v\right]. \end{aligned}$$

As $\mathcal{S}^{\mathcal{H}}$ is parallel, we must have that $R^{\nabla^{nc},\mathcal{S}}(X,Y)\Theta = f\cdot\Theta$ for some function $f = f_{X,Y} : M \to \mathbb{R}$. However, using the explicit form of the real Clifford representation from Remark 1.5, it is a straightforward algebraic calculation that for $i<j$ we have $e_i\cdot e_j\cdot v \propto v$ iff $(i,j) = (2k-1,2k)$ for $k = 1,...m+1$, and in this case we have $e_{2k-1}\cdot e_{2k}\cdot v = v$. Consequently,

$$R^{\nabla^{nc},\mathcal{S}}(X,Y)\Theta = -\frac{1}{2}\sum_{j=1}^{m+1}\langle R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)s_{2j-1},s_{2j}\rangle_{\mathcal{T}}\cdot\Theta. \tag{3.41}$$

On the other hand, as ∇^{nc} is metric wrt. $\langle\cdot,\cdot\rangle_{\mathcal{T}}$, we have that $\langle R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)s_i,s_j\rangle_{\mathcal{T}} = -\langle R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)s_j,s_i\rangle_{\mathcal{T}}$ from which follows that

$$R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)(s_{2j-1}+s_{2j}) = 2\cdot\langle R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)s_{2j-1},s_{2j}\rangle_{\mathcal{T}}(s_{2j-1}+s_{2j}) + \sum_{k\neq j} r_k(s_{2k-1}+s_{2k})$$

[14] Note that with these choices, in split signature this representation descends to a real representation

for some $r_k \in \mathbb{R}$. Hence,

$$\text{tr } R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)_{|\mathcal{H}} = 2 \cdot \sum_{j=1}^{m+1} \langle R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)s_{2j-1}, s_{2j}\rangle_{\mathcal{T}}.$$

Thus, together with (3.41) we have to show that

$$\text{tr } R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)_{|\mathcal{H}} = 0 \Leftrightarrow \text{tr } W^g(X,Y)_{|L} = 0. \tag{3.42}$$

We will actually show that these two traces are always equal: Proposition 3.32 yields the existence of $\widetilde{M} \subset M$ such that for every $x \in \widetilde{M}$, there is $g \in c_{|\widetilde{M}}$[15] such that at x we have

$$\mathcal{H} \overset{g}{=} \text{span}\left(\begin{pmatrix} 0 \\ K_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ K_m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \tag{3.43}$$

for lightlike vectors $K_i \in T_xM$ which together span $L = \text{pr}_{TM}(\mathcal{H} \cap \mathcal{I}^{\perp})$. The g–metric representation (2.26) of $R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)$ yields as $\mathcal{H}$ is parallel that at x

$$R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)\begin{pmatrix} 0 \\ K_j \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ W^g(X,Y)K_j \\ 0 \end{pmatrix} \text{ and } R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ C^g(X,Y)^{\sharp} \\ 0 \end{pmatrix}.$$

Thus, written as a matrix wrt. the g–metric basis vectors of $\mathcal{H}(x) \overset{g}{=} L_x \oplus span \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ this reads as

$$R^{\nabla^{nc},\mathcal{T}(M)}(X,Y)_{|\mathcal{H}} = \begin{pmatrix} W^g(X,Y)_{|L} & C^g(X,Y)^{\sharp} \\ 0 & 0 \end{pmatrix}.$$

From this, (3.42) follows immediately. Thus there is by the preceding remarks a parallel spin tractor ψ and it is clear from the construction that $\mathcal{H} = \ker \psi$.
We prove the further assertions. It is clear by Proposition 1.19 that if $\psi_{1,2}$ are spin tractors with $\ker \psi_{1,2} = \mathcal{H}$ we must have that $\psi_1 = f \cdot \psi_2$ for some smooth function f without zeroes. If $\psi_{1,2}$ are parallel, f has to be constant. Thus, ψ is unique up to multiplication with nonzero constants. Moreover, Proposition 3.28 implies that there is locally on $\widetilde{M}$ a metric g such that $\varphi = \widetilde{\Phi}^g\,(proj^g_+\psi)$ is a real, parallel pure spinor. By Theorem 3.36 this leads to a local normal form for g. If conversely $(M^{m,m}, c)$ admits a parallel pure spin tractor, it is clear that $\mathcal{H} = \ker \psi$ satisfies 1. and 2. follows from (3.40). □

Remark 3.39 A slight modification of the above proof reveals that if in the notation of Theorem 3.37 we impose only the first condition on $\mathcal{H}$ but not the tr W^g-condition, then there is a recurrent spin tractor $\psi \in \Gamma(\widetilde{M}, \mathcal{S})$, i.e. $\nabla^{nc}_X \psi = \theta(X) \cdot \psi$ for a 1-form $\theta \in \Omega^1(\widetilde{M})$ and one finds locally a metric g such that $\varphi = \widetilde{\Phi}^g(proj^g_+\psi) \in \Gamma(M, S^g)$ is recurrent on (M,g) as well. Such spinor fields have been studied recently in [Gal13].

[15] In the proof of Proposition 3.32 we constructed an explicit local conformal change $e^{2\sigma}$ around x in order to obtain locally around x the desired metric. We can simply multiply σ by a cut-off-function in order to obtain a metric which is defined on all of $\widetilde{M}$.

Remark 3.40 Note that every parallel (pointwise) pure spin tractor ψ on (M,c) of signature (m,m) does not only give rise to a $(m+1)$-dimensional totally lightlike and parallel distribution ker $\psi \subset \mathcal{T}(M)$ but by Lemma 1.24 also to a totally lightlike and decomposable twistor $(m+1)$–form $\alpha := \alpha_{\psi}^{m+1} \in \Omega_{\mathcal{T}}^{m+1}(M)$. One can now also use the nc-Killing form theory, concretely Lemma 3.28 to deduce that there is a local metric $g \in c$ such that $proj^g_{\Lambda,+}\alpha = \alpha^m_{\varphi}$ is parallel and from this it is easy to deduce that the spinor itself is parallel. This gives an alternative proof of Proposition 3.8 for this special case.

It remains to prove the curvature formula used in the proof of Theorem 3.37:

Proposition 3.41 *For a pseudo-Riemannian conformal spin manifold (M^n,c), the curvatures $R^{\nabla^{nc},\mathcal{S}}$ of $(\mathcal{S}(M),\nabla^{nc})$ and $R^{\nabla^{nc},\mathcal{T}}$ of $(\mathcal{T}(M),\nabla^{nc})$ are related by*

$$R^{\nabla^{nc},\mathcal{S}}(X,Y)\Theta = \frac{1}{2}\sum_{1\leq i<j\leq n+2} \epsilon_i\epsilon_j \langle R^{\nabla^{nc},\mathcal{T}}(X,Y)s_i,s_j\rangle_{\mathcal{T}} s_i \cdot s_j \cdot \Theta,$$

where $X,Y \in \mathfrak{X}(M)$, $\Theta \in \Gamma(\mathcal{S}(M))$ and $(s_1,...,s_{n+2})$ is any local frame in $\mathcal{T}(M)$ which is pseudo-orthonormal wrt. $\langle\cdot,\cdot\rangle_{\mathcal{T}}$.

Proof. We begin with recalling some general facts regarding the curvature of principal bundle connections (cf. [Bau09]): Let $(\mathcal{P},\pi,M;G)$ be any principal bundle with structure group G and connection $\omega \in \Omega^1(\mathcal{P},\mathfrak{g})$ with curvature $F^\omega = d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(\mathcal{P},\mathfrak{g})$. Here, if $\omega = \sum_{i=1}^r \omega_i a_i$ for a basis $(a_1,...,a_r)$ of $\mathfrak{g}$ and $\omega_i \in \Omega^1(\mathcal{P})$, then the bracket is given by

$$[\omega,\omega] := \sum_{i,j}(\omega^i \wedge \omega^j)[a_i,a_j]_{\mathfrak{g}} \in \Omega^2(\mathcal{P},\mathfrak{g}).$$

Assume that $\rho : G \to GL(V)$ is a representation of G over a vector space V and consider the associated vector bundle $E := \mathcal{P}\times_{(G,\rho)} V$ with induced covariant derivative $\nabla := \nabla^\omega$: $\Gamma(E) \to \Gamma(T^*M \otimes E)$ and curvature endomorphism $R^\nabla(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \in End(E)$ for $X,Y \in \mathfrak{X}(M)$. Fix $x \in M$. For arbitrary $p \in \mathcal{P}_x$ and associated fibre isomorphism $[p] : V \to E_x, v \mapsto [p,v]$, the curvatures $R^\nabla(X,Y)$ and F^ω are related by

$$R^\nabla_x(X,Y) = [p] \circ \rho_* \left(F^\omega_p(X^\omega,Y^\omega)\right) \circ [p]^{-1}. \tag{3.44}$$

Here, for $X \in \mathfrak{X}(M)$, the vector field $X^\omega \in \mathfrak{X}(\mathcal{P})$ denotes the horizontal lift of X wrt. ω which is uniquely determined by the requirements $X^\omega(p) \in \ker\,\omega(p)$ and $d\pi_p(X^\omega(p)) = X(\pi(p))$ for all $p \in \mathcal{P}$.

We now apply this notion to our original setting, i.e. $(\mathcal{P},\pi,M;G)$ being one of the principal bundles $(\overline{\mathcal{P}}^1_+,\pi_{\mathcal{P}},M;SO^+(p+1,q+1))$ or the $(\overline{f}^1,\lambda)$ reduction $(\overline{\mathcal{Q}}^1_+,\pi_{\mathcal{Q}},M;Spin^+(p+1,q+1))$, and principal bundle connections $\overline{\omega}^{nc}$ and $\overline{\overline{\omega}}^{nc}$, respectively. We first claim that in this setting

$$F^{\overline{\overline{\omega}}^{nc}}(Z_1,Z_2) = \lambda_*^{-1}\left(F^{\overline{\omega}^{nc}}(d\overline{f}^1(Z_1),d\overline{f}^1(Z_2))\right)$$

for $Z_1,Z_2 \in \mathfrak{X}(\overline{\mathcal{Q}}^1_+)$. To this end, let a_i be a basis of $\mathfrak{so}(p+1,q+1)$ such that $\overline{\omega}^{nc} = \sum_{i=1}^r W_i a_i$ for 1-forms $W_i \in \Omega^1(\overline{\mathcal{P}}^1_+)$. It follows together with

$$\overline{\overline{\omega}}^{nc} = \lambda_*^{-1} \circ \overline{\omega}^{nc} \circ d\overline{f}^1 \tag{3.45}$$

that

$$
\begin{aligned}
F^{\overline{\widetilde{\omega}}^{nc}}(Z_1,Z_2) &= d\left(\lambda_*^{-1}\circ\overline{\omega}^{nc}\circ d\overline{f}^1\right)(Z_1,Z_2)+\frac{1}{2}\left[\lambda_*^{-1}\left(\overline{\omega}^{nc}(d\overline{f}^1)\right),\lambda_*^{-1}\left(\overline{\omega}^{nc}(d\overline{f}^1)\right)\right]_{\mathfrak{spin}}(Z_1,Z_2)\\
&= \lambda_*^{-1}\left(d(\overline{\omega}^{nc})\left(d\overline{f}^1(Z_1),d\overline{f}^1(Z_2)\right)\right)+\frac{1}{2}\sum_{i,j}W^i(d\overline{f}^1(Z_1))\cdot W^j(d\overline{f}^1(Z_2))\cdot\lambda_*^{-1}\left([a_i,a_j]_{\mathfrak{so}}\right)\\
&= \lambda_*^{-1}\left(F^{\overline{\omega}^{nc}}(d\overline{f}^1(Z_1),d\overline{f}^1(Z_2))\right).
\end{aligned}
$$

We now let $x\in M$ and $u\in\overline{\mathcal{Q}}^1_x$ such that $s_i(x)=\left[\overline{f}^1(u),e_i\right]\in\mathcal{T}_x(M)$. Further, consider a spinor $\Theta\in\mathcal{S}_x(M)$. Then there is $v\in\Delta_{p+1,q+1}$ with $\Theta=[u,v]$. Let $X,Y\in\mathfrak{X}(M)$ and let $X^*\in\mathfrak{X}(\overline{\mathcal{P}}^1_+)$ denote the horizontal lift of X wrt. $\overline{\omega}^{nc}$ and $\widetilde{X}^*\in\mathfrak{X}(\overline{\mathcal{Q}}^1_+)$ the horizontal lift wrt. $\overline{\widetilde{\omega}}^{nc}$. As a direct consequence of (3.45), we have that $X^*=d\overline{f}^1\left(\widetilde{X}^*\right)$. Finally, let $\rho: Spin(p+1,q+1)\to\Delta_{p+1,q+1}$ denote the standard spinor representation. We find constants F_{ij} such that

$$F^{\overline{\omega}^{nc}}_{\overline{f}^1(u)}(X^*,Y^*)=\sum_{i<j}F_{ij}\cdot E_{ij}\in\mathfrak{so}(p+1,q+1).$$

Putting all this together, yields by definition that

$$
\begin{aligned}
R^{\nabla^{nc},\mathcal{S}}(X,Y)\Theta &= [u]\left(\rho_*\left(F_u^{\overline{\widetilde{\omega}}^{nc}}\left(\widetilde{X}^*,\widetilde{Y}^*\right)\right)v\right)\\
&= [u]\left(\rho_*\left(\lambda_*^{-1}\left(F^{\overline{\omega}^{nc}}_{\overline{f}^1(u)}(d\overline{f}^1(\widetilde{X}^*),d\overline{f}^1(\widetilde{Y}^*))\right)\right)v\right)\\
&= [u]\left(\rho_*\left(\sum_{i<j}\lambda_*^{-1}(F_{ij}E_{ij})\right)(v)\right)\\
&= \left[u,\frac{1}{2}\sum_{i<j}F_{ij}\cdot e_i\cdot e_j\cdot v\right]=\frac{1}{2}\sum_{i<j}F_{ij}s_i\cdot s_j\cdot\Theta.
\end{aligned}
$$

In order to determine the F_{ij}, we again apply (3.44) to obtain

$$
\begin{aligned}
R^{\nabla^{nc},\mathcal{T}}(X,Y)s_i &= \left[\overline{f}^1(u)\right]\left(F^{\overline{\omega}^{nc}}_{\overline{f}^1(u)}(d\overline{f}^1(\widetilde{X}^*),d\overline{f}^1(\widetilde{Y}^*))e_i\right)\\
&= \left[\overline{f}^1(u),\sum_{k<l}F_{kl}E_{kl}e_i\right]=\left[\overline{f}^1(u),\epsilon_i\sum_j F_{ij}e_j\right],
\end{aligned}
$$

which immediately implies that $\langle R^{\nabla^{nc},\mathcal{T}}(X,Y)s_i,s_j\rangle_{\mathcal{T}}=\epsilon_i\epsilon_jF_{ij}$ and inserting this proves the Proposition. □

Remark 3.42 Clearly, Proposition 3.41 is the conformal analogue to (2.7), which relates the curvatures of ∇^g and ∇^{S^g} by a resembling formula. If one applies Proposition 3.41 to $\psi\in Par(\mathcal{S}(M),\nabla^{nc})$, one recovers the integrability conditions $W^g(X,Y)\cdot\varphi=0$ and $W^g(X,Y)\cdot D^g\varphi=nC^g(X,Y)\cdot\varphi$ for $g\in c$, $\varphi=\widetilde{\Phi}^g(proj^g_+\psi)\in\ker P^g$ and $X,Y\in TM$ from Proposition 2.11.

3.5 A partial classification result

We now apply and summarize the previous constructions and statements in order to obtain the following partial classification result for conformal structures in arbitrary signature admitting twistor spinors expressed in terms of conformal holonomy:

Theorem 3.43 *Let $\psi \in Par(\mathcal{S}(M), \nabla^{nc})$ be a parallel spin tractor on a conformal spin manifold (M,c) of signature (p,q) and dimension $n = p+q \geq 3$. For $g \in c$ let $\varphi = \widetilde{\Phi}^g(proj^g_+\psi) \in ker P^g$ denote the associated twistor spinor. Exactly one of the following cases occurs:*

1. *It is $ker\, \psi \neq \{0\}$. In this case, φ can locally be rescaled to a parallel spinor on an open, dense subset $\widetilde{M} \subset M$, and $ker\, \varphi \subset TM$ is an integrable distribution on $\widetilde{M}$. In case that the respective metric holonomy acts irreducible and the space is not locally symmetric, it is one of the list in Theorem 3.2. Otherwise, one has a parallel spinor on a Ricci-isotropic pseudo Walker manifold. The conformal holonomy representation $Hol(M,c)$ is never irreducible but fixes a nontrivial totally lightlike subspace.*

2. *It is $ker\, \psi = \{0\}$. The spinor φ cannot be locally rescaled to a parallel spinor. Depending on the conformal holonomy representation, exactly one of the following cases occurs:*

   a) *$Hol(M,c)$* fixes a totally lightlike subspace. *In this case, there is locally around each point a metric in the conformal class such that φ is a twistor spinor which is not Killing on a Ricci-isotropic pseudo Walker manifold. If $Hol(M,c)$ fixes an isotropic line, then there is a Ricci-flat metric $g \in c$ on which $D^g\varphi$ is nontrivial and parallel.*

   b) *$Hol(M,c)$* acts reducible and fixes only nondegenerate subspaces. *In this case, there is around each point of an open, dense subset $\widetilde{M} \subset M$ an open neighbourhood U and a metric $g \in c_U$ such that either*

      - *(U,g) is an Einstein space with $scal^g \neq 0$ and φ decomposes into the sum of two Killing spinors.*
      - *$(U,g) \overset{isom.}{\cong} \pm dt^2 \times (V,g')$, where the last factor is an Einstein space with $scal^g \neq 0$ admitting a Killing spinor.*
      - *$(U,g) \overset{isom.}{\cong} (H,h) \times (V,g')$, where the first factor is a two dimensional space and (V,g') is as above.*
      - *$(U,g) \overset{isom.}{\cong} (M_1,g_1) \times (M_2,g_2)$, where (M_i,g_i) are Einstein spaces of dimensions ≥ 3. (M_1,g_1) admits a real Killing spinor to the Killing number $\lambda \neq 0$ and (M_2,g_2) admits an imaginary Killing spinor to $i \cdot \mu$, where $|\lambda| = |\mu|$.*

   c) *$Hol(M,c)$* acts irreducible. *If in addition the action on the conformal Möbius sphere is transitive, then $Hol(M,c)$ is one of the groups listed in Theorem 3.13. If there exists a metric $g \in c$ satisfying both $C^g = 0$ and $\nabla W^g \neq 0$, i.e. (M,g) is a Cotton space and not conformally symmetric, then $Hol(M,c)$ is also one of the groups in Theorem 3.13.*

Proof. We have already proved 1. in Proposition 3.28 and the first part of 2.a) follows directly from Theorem 3.22. The fact that the spinor cannot be a Killing spinor in this situation follows since a manifold admitting a Killing spinor always has nonvanishing scalar curvature. Moreover, if ker $\psi = \{0\}$ and $Hol(M,c)$ fixes an isotropic line, we have (at least locally) a Ricci-flat metric g in the conformal class admitting a nonparallel twistor spinor. It follows directly from $K^g = 0$ that $D^g\varphi$ is parallel.
We now prove 2.b): By assumption, there is a $Hol(M,c)$-invariant, nondegenerate subspace $E \subset \mathbb{R}^{p+1,q+1}$. If dim $E = 1$, it follows from Theorem 3.16 that there is locally an Einstein scale g with scal$^g \neq 0$ in the conformal class. [Boh99] shows that every twistor spinor on an Einstein space decomposes into the sum of two Killing spinors.
If dim E=2, we have by Theorem 3.17 that there is around each point a local metric splitting $\pm dt^2 \times (V,g')$ in the conformal class where the last factor is an Einstein space. Clearly, the spin structure on (M,g) induces a spin structure on (V,g'). The well-known formulas for the restriction of spin structures to hypersurfaces (cf. [BGM05]) yield for this situation that $\left(\nabla^{M,S^g}_X \varphi\right)_{|V} = \nabla^{V,S^{g'}}_X \varphi_{|V}$ for $X \in TV$, where the spinor bundles of (M,g) and (V,g') are suitably identified along V, which immediately yields that $\varphi_{|V}$ is a twistor spinor on the Einstein manifold (V,g') which therefore again decomposes into the sum of Killing spinors.
For the case dim E=3 one uses the same argument and the formula for induced spinor derivatives in codimension 2 from [Lei01]. More precisely, Theorem 3.17 yields that there is a local metric g on $U \subset M$ in the conformal class and a splitting into $(U,g) = (H,h)\times(V,g')$ in the conformal class, where the second factor is an Einstein space. By restricting U to a smaller set, if necessary, we may assume that TH can be trivialized by a h-orthonormal frame field. Then there is a naturally induced spin structure on (V,g'). Moreover, one can identify $S^g_{|V} \cong S^{g'} \oplus S^{g'}$, and under this identification the spinor derivative behaves for $\varphi = \varphi_1 + \varphi_2$ as (cf. [Lei01])

$$\nabla^{S^g}_X \varphi_{|V} = \nabla^{S^{g'}}_X \varphi_1 + \nabla^{S^{g'}}_X \varphi_2,$$

where $X \in TV$. If $\varphi = \varphi_1 + \varphi_2$ is the twistor spinor on (U,g), the spinor $\phi = \phi_1 + \phi_2 := g(X,X)\cdot X\cdot\nabla^{S^g}_X\varphi$ does not depend on the choice of $X \in TU$ with $g(X,X) = \pm 1$. It follows that $\psi_i = g'(X,X)\cdot X\cdot\nabla^{S^{g'}}_X \varphi_i$ does not depend on the choice of $X \in TV$ with $g'(X,X) = \pm 1$, i.e. φ_i are (possibly linearly dependent) twistor spinors on V for $i = 1,2$.
If dim $E > 3$, we may also assume that dim $E^\perp > 3$, as otherwise one of the previous cases applies to $E^\perp$. The local metric splitting into Einstein spaces $M_1 \times M_2$ is then a direct consequence of Theorem 3.17. From the holonomy formula of the same Theorem we obtain that there are parallel spinors on the cones $C(M_i)$ (also cf. [Lei07]): To see this, one uses that the lift of $Hol(M,c)$ to $Spin^+(p+1,q+1)$ is contained in the product of the spin groups with signatures of the two cones $C(M_i)$. [Lei04] shows that in such an algebraic situation there is a fixed spinor in $\Delta_{p+1,q+1}$ iff both factors admit a fixed spinor. These correspond to Killing spinors on the base manifolds M_i by a well-known construction from Bär, cf. [Bär93]. Theorem 3.17 relates the scalar curvatures of the Einstein factors and by the well-known relation scal$^g = 4n(n-1)\lambda^2$ between Killing number λ and scalar curvature, the claim regarding the Killing numbers follows.
It remains to consider the case of irreducibly acting conformal holonomy. The situation where the action on the conformal Möbius sphere is transitive was already studied in Theorem 3.13. [AL06] studies an ambient metric construction which under the condition

$C^g = 0$ for some $g \in c$ associates a manifold $(\widetilde{M}, h)$ of signature $(p+1, q+1)$ to (M,c) which - under a canonical identification $T\widetilde{M} \cong \mathcal{T}(M)_g$ - satisfies that $Hol(M,c) = Hol(\widetilde{M}, h)$. It follows that $(\widetilde{M}, h)$ admits a parallel spinor iff (M,g) admits a twistor spinor, and $(\widetilde{M}, h)$ has irreducibly acting holonomy as this is assumed for (M,c). Moreover the curvature formulas from [AL06] directly yield that $(\widetilde{M}, h)$ is not locally symmetric in case $\nabla^g W^g \neq 0$. Thus, one can apply Theorem 3.2 to $Hol(M,c) = Hol(\widetilde{M}, h)$. It follows from the list in Theorem 3.2 that $Hol(M,c)$ is either one of the groups which have already appeared for the transitive case in Theorem 3.13 or it is $G_2^{\mathbb{C}} \subset SO^+(7,7)$ or $Spin(7)^{\mathbb{C}} \subset SO^+(8,8)$. However, [Alt08] shows that the last two groups do not appear as conformal holonomy groups. □

Remark 3.44 A generation principle for geometries of type $2.b)$ admitting twistor spinors is discussed in [Lei07]: One takes a Riemannian Einstein space (M^1, g_1) of positive scalar curvature admitting a real Killing spinor φ_1 to the Killing number λ (cf. [Bär93]) and a negative-definite Einstein space (M^2, g_2) admitting an imaginary Killing spinor φ_2 to the Killing number $-i\lambda$. Then, under appropriate identification of the spinor bundles (cf. [Lei04]), the spinor $\varphi_1 \otimes \varphi_2$ is a zero-free twistor spinor for $(M_1 \times M_2, [g_1 \times g_2])$. Furthermore, the twistor equation on products $(M \times \mathbb{R}, g + dt^2)$, where (M,g) is an Einstein space with $\text{scal}^g \neq 0$ admitting Killing spinors has been discussed in [BFGK91].
Moreover, we have already discussed geometries of type $2.c)$ in Remark 3.14. Thus, all these cases from Theorem 3.43 really occur as local geometries admitting twistor spinors. Geometries occurring in the last case of Theorem 3.43, i.e. those with $C^g = 0$ are so called conformal C-spaces which are also of interest in physics and have been studied in [GN07, Lei06].

The interesting new case compared to previous classification results for the Riemannian and Lorentzian case is the situation 2.a): There is (locally) a Ricci-isotropic pseudo-Walker metric in the conformal class admitting a twistor spinor which cannot be rescaled to a parallel spinor. Let us first consider a non-trivial example for this situation:

Example 3.45 For $n_1 = p + q_1$ even let $H^{p,q_1} := \{x \in \mathbb{R}^{p+1,q_1} \mid \langle x, x\rangle_{p+1,q_1} = -1\}$ equipped with the induced metric g_1 be the pseudo-Riemannian model-space of constant negative sectional curvature for signature (p, q_1). Furthermore, let (M_2, g_2) be a odd-dimensional complete, non-conformally flat and simply-connected Riemannian spin manifold of dimension q_2 carrying a real Killing spinor to Killing number $\frac{1}{2}$, cf. [Bär93] for a classification. We set

$$(M,g) = (H^{p,q_1} \times M_2, g_1 \times g_2).$$

The first factor admits imaginary Killing spinors to the Killing numbers $\pm\frac{i}{2}$. Thus, Theorem 3.17 applies, yielding that

$$\begin{aligned} Hol(M,[g]) = \{Id\} \times Hol(C(M_2), g_{2,C}) &\subset SO^+(p+1, q_1) \times SO^+(0, q_2+1) \\ &\subset SO^+(p+1, q_1+q_2+1), \end{aligned} \tag{3.46}$$

where $(C(M_2), g_{2,C})$ denotes the metric cone over (M_2, g_2). In particular, (M,c) is not conformally flat, as otherwise $M_2 \cong S^{q_1}$. It follows that

$$Hol(\mathcal{S}(M), \nabla^{nc}) \subset Spin^+(p+1, q_1) \times Spin^+(0, q_2+1),$$

which acts on Δ_{p+1,q_1+q_2+1}. However,

$$\Delta_{p+1,q_1+q_2+1|Spin^+(p+1,q_1)\times Spin^+(0,q_2+1)} \cong \Delta_{p+1,q_1} \otimes \Delta_{0,q_2+1},$$

which can be seen as follows: Let $\rho_1 : Cl(p+1,q_1) \to \Delta_{p+1,q_1}$ and $\rho_2 : Cl(0,q_2+1) \to \Delta_{0,q_2+1}$ be irreducible complex representations. As q_2+1 is even, the complex volume element $\omega_{\mathbb{C}} \in Cl(0,q_2+1)$ squares to 1 and anticommutes under Clifford multiplication with every vector in $\mathbb{R}^{q_2+1}$. We set

$$\Phi : \mathbb{R}^{p+1,q_1+q_2+1} = \mathbb{R}^{p+1,q_1} \times \mathbb{R}^{0,q_2+1} \to GL(\Delta_{p+1,q_1} \otimes \Delta_{0,q_2+1}),$$

uniquely determined by

$$\Phi((X,Y))(v_1 \otimes v_2) = (X+Y)\cdot(v_1\otimes v_2) := (\rho_1(X)(v_1)) \otimes \omega_{\mathbb{C}} \cdot v_2 + v_1 \otimes (\rho_2(Y)(v_2)). \tag{3.47}$$

Φ extends to an irreducible representation of $Cl(p+1,q_1+q_2+1)$.
Let us now specialize the situation to the case $p=q_1=3$. The parallel spinors on the flat timelike cone over $H^{3,3}$ are in one-to-one-correspondence to spinors in $\Delta^{\mathbb{C}}_{4,3}$. The latter spinor module has been investigated in [Kat99]. It follows that there exists a parallel spinor ψ_1 on the cone over $H^{3,3}$ with the additional property $X\cdot\psi \not\sim \psi$ for $X \in TC(H^{3,3})\backslash\{0\}$ because the reference shows on an algebraic level that such a spinor exists in $\Delta^{\mathbb{C}}_{4,3}$.
Furthermore, the real Killing spinor on (M_2,g_2) corresponds to a parallel spinor ψ_2 on $(C(M_2),g_{2,C})$. It follows with the above identifications that $\psi := \psi_1 \otimes \psi_2 \in \Gamma(M,\mathcal{S}(M))$ is a parallel spin tractor. Let $(X,Y) \in \mathcal{T}(M) \cong TC(H^{3,3}) \oplus TC(M_2)$. By (3.47) we have that $(X,Y)\cdot\psi = 0$ implies that $X\cdot\psi_1 = \alpha\cdot\psi_1$ for some α which is not possible by choice of ψ_1 unless $X=0$. It follows that ker $\psi=\{0\}$. That is, the spin tractor ψ does not give rise to a parallel spinor for any local metric in the conformal class.
Moreover, by (3.46), $Hol(M,[g])$ fixes a 3-dimensional totally lightlike subspace. Whence, there is at least locally a Ricci-isotropic pseudo-2-Walker metric $\widetilde{g}\in[g]$ admitting a twistor spinor $\varphi = \widetilde{\Phi}^{\widetilde{g}}(proj_+\psi)$ which is not conformally equivalent to a parallel spinor.

Remark 3.46 We do not know whether there are examples of Ricci-isotropic pseudo-Walker manifolds admitting non-parallel twistor spinors whose conformal holonomy representation does not fix any non-degenerate subspace, i.e. which are in contrast to Example 3.45 not conformally equivalent to products of Einstein spaces.

There is a further consequence for the situation 2.a) in Theorem 3.43:

Proposition 3.47 *Let (M^n,g) be a Ricci-isotropic pseudo-Walker spin manifold with parallel totally lightlike, parallel distribution $L\subset TM$, $Ric(TM)\subset L$ of rank $k>1$ admitting a non-parallel twistor spinor φ. We fix a local basis $l_1,...,l_k$ of L and let $\omega := l_1^\flat \wedge....\wedge l_k^\flat$. Then $\phi := \omega\cdot D^g\varphi$ is a -possibly trivial- recurrent spinor.*

Proof. As L is parallel, there is a 1-form θ such that $\nabla^g\omega = \theta\otimes\omega$. We calculate that

$$\nabla^{S^g}_X\phi = \theta(X)\cdot\phi - \omega\cdot\frac{n}{2(n-2)}\cdot\underbrace{Ric^g(X)}_{\in L}\cdot\varphi.$$

The second summand vanishes because as L is totally lightlike we obtain $\omega\cdot l_i = \omega\wedge l_i^\flat = 0$ for all $1\le i\le k$. Thus, ϕ is recurrent. □

We sum up our discussion as follows: Let $\psi \in \Gamma(M, \mathcal{S})$ be a parallel spin tractor on a conformal spin manifold (M, c) with associated twistor spinor $\varphi \in \Gamma(S^g)$ for $g \in c$. One of the cases in the following table occurs:

ker ψ	$Hol(M,c)$	local geometry $g \in c$, behaviour of φ
$\neq \{0\}$	fixes ker ψ	φ parallel on Ricci-isotropic pseudo-Walker metric
$\{0\}$	fixes totally lightlike subspace H	φ non-parallel on Ricci-isotropic pseudo-Walker metric
	• dim H=1	$D^g\varphi$ parallel
	• dim $H > 1$	$\omega \cdot D^g\varphi$ recurrent
	fixes only non-degenerate subspaces	Splitting into Einstein spaces admitting Killing spinors
	acts irreducibly on $\mathbb{R}^{p+1,q+1}$	
	• acts transitively on $Q^{p,q}$ or there is a non-conformally symmetric C-space in the conformal class	Fefferman spin space or S^3-bundle over quat. contact manifold with non-parallel twistor spinors, or generic cases in signatures $(3,2)$,$(3,3)$
	• does not act transitively on $Q^{p,q}$ and there is no C-space in the conformal class	*No example known*

Remark 3.48 Theorem 3.43 and the subsequent discussion show that in order to obtain a complete classification of conformal spin manifolds admitting twistor spinors which are not locally conformally equivalent to parallel spinors, it remains to do the following:

1. Classify Ricci-isotropic, non-Ricci-flat pseudo-Walker manifolds admitting non-parallel twistor spinors. This geometric structure seems to be rather special.

2. Are there conformal geometries (M, c) admitting twistor spinors spinors which are no non-conformally symmetric conformal C-spaces and $Hol(M, c)$ acts irreducibly on $\mathbb{R}^{p+1,q+1}$, but not transitively on the Möbius sphere (*the case not covered by 2.c)*)?

As we have seen, the most interesting case of *true* twistor spinors, i.e. those which cannot be rescaled to parallel spinors locally, correspond from a conformal holonomy point of view to spin tractors ψ with ker $\psi = \{0\}$. From an algebraic point of view, i.e. to understand the conditions for the conformal holonomy which arises from this case, one therefore has to answer the following question:
Let $v \in \Delta_{p+1,q+1}$ be a spinor such that the map $\mathbb{R}^{p+1,q+1} \ni X \mapsto X \cdot v \in \Delta_{p+1,q+1}$ is injective, i.e. ker $v = 0$. What properties does this imply for the λ-image of its stabilizer, $\lambda(Stab_v Spin^+(p+1, q+1)) \subset SO^+(p+1, q+1)$?
One could hope that there are common properties of these $\lambda(Stab_v Spin^+(p+1, q+1))$-groups and that there is a finite classification list for such groups. One could then try to distinguish those which can really occur as conformal holonomy groups.

4 The Zero Set of a Twistor Spinor

When dealing with twistor spinors and constructing examples, the case of a twistor spinor admitting a zero always turns out to be the most involved situation. For $\varphi \in \Gamma(S^g)$ a twistor spinor on $(M^{p,q}, g)$, we let Z_φ denote its zero set. Clearly, by Lemma 2.35 this set is invariant under a conformal change of g. The following two questions are of interest in this situation:

1. What are the possible shapes of Z_φ ?
2. What are possible local geometries off the zero set. In particular, is there a relation between the shape of Z_φ and the conformal geometry of M off the zero set ?

Up to now only little is known about the answer of these two question in an arbitrary pseudo-Riemannian setting. We present a short overview:

Proposition 4.1 ([BFGK91]) *On a Riemannian manifold, the zero set of a twistor spinor consists of a countable union of isolated points. Moreover, if $Z_\varphi \neq \varnothing$, then off the zero set the metric can be rescaled such that the spinor is parallel.*

The proof is basically a direct calculation and based on the consideration of the Hessian of $||\varphi||^2$ at a zero. Moreover, [Hab94] studies under which conditions Z_φ is an isolated point. In the Lorentzian case, the situation is already more involved. As $Z_\varphi = Z_{V_\varphi}$ in this situation, one can find the geometric structure of Z_φ by studying the zero set of certain conformal vector fields as done in [Lei01]. Together with aspects of the nc-Killing form theory, this leads to the following description in [Lei09]:

Theorem 4.2 *Let $\varphi \in \Gamma(S^g)$ be a twistor spinor with $Z_\varphi \neq \varnothing$ on a Lorentzian spin manifold (M, g). Then Z_φ consists either of*

1. *isolated points and off a singular set, the metric g is locally conformally equivalent to a static monopole $-dt^2 + h$ where h is a Riemannian metric with parallel spinor, or*
2. *isolated images of null geodesics and off the zero set the metric g is locally equivalent to a Brinkmann metric with parallel spinor.*

We now address the problem of finding the shape of Z_φ in arbitrary pseudo-Riemannian signature. Using tractor methods, we prove in this chapter:

Theorem 4.3 *Let $\varphi \in \Gamma(M^{p,q}, S^g)$ be a twistor spinor with zero x. Then the zero set Z_φ is an embedded, totally geodesic and totally lightlike submanifold of M whose dimension equals dim ker $D^g\varphi(x)$, where the last quantity does not depend on the choice of $x \in Z_\varphi$.*

Moreover, for every $x \in Z_\varphi$ there are open neighbourhoods U of x in M and V of 0 in T_xM such that

$$Z_\varphi \cap U = exp_x\left(ker\ D^g\varphi(x) \cap V\right). \tag{4.1}$$

Remark 4.4 Note that Theorem 4.3 completely includes the results concerning the shape of Z_φ from Proposition 4.1 for the Riemannian case, where we always have that ker $D^g\varphi(x) = \{0\}$, and Theorem 4.2 for the Lorentzian case, where dim ker $D^g\varphi(x) \in \{0,1\}$ which distinguishes the two cases from the Theorem.

We will first focus on the proof of Theorem 4.3 and then study further implications. For the proof, we first study the zero set of twistor spinors on the homogeneous model and then relate this to the zero set of arbitrary twistor spinors using the curved orbit decomposition for Cartan geometries.

4.1 Zeroes of twistor spinors on the homogeneous model

We completely describe the *global* structure of the zero set of twistor spinors on the homogeneous model $\widehat{C}^{p,q} = (\widehat{Q}^{p,q},[g_{st}])$. Using (2.18) we identify $\widehat{Q}^{p,q}$ with the product $S^p \times S^q \subset \mathbb{R}^{p+1,q+1}$, and it is equipped with the conformally flat standard metric $g_{St} := -g_{S^p} + g_{S^q}$. We follow [Lei01] in order to construct all twistor spinors on $\widehat{C}^{p,q}$:
Every $x \in \mathbb{R}^{n+2} \cong \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ decomposes into $x = (x_1,x_2)$. There is a natural, globally defined orthonormal frame field on the normal bundle $N\widehat{Q}^{p,q} \subset \mathbb{R}^{p+1,q+1}$, given by $\zeta_0(x) = (x_1,0)$ and $\zeta_{n+1}(x) = (0,x_2)$ for $x \in \widehat{Q}^{p,q}$. The spin structure on $\widehat{C}^{p,q}$ is then naturally induced by the standard spin structure on $\mathbb{R}^{p+1,q+1}$, and the spinor bundles are related by

$$S^{\mathbb{R}^{p+1,q+1}}_{|\widehat{Q}^{p,q}} \cong \underbrace{Ann\left(\zeta_0+\zeta_{n+1}\right)}_{\cong S^{\widehat{Q}^{p,q},g_{St}}} \oplus \underbrace{Ann\left(\zeta_0-\zeta_{n+1}\right)}_{\cong S^{\widehat{Q}^{p,q},g_{St}}}, \tag{4.2}$$

where $Ann\left(\zeta_0 \pm \zeta_{n+1}\right) = \{v \in \Delta_{p+1,q+1} \mid (\zeta_0 \pm \zeta_{n+1})\cdot v = 0\}$. Wrt. this splitting, every spinor φ on $\mathbb{R}^{p+1,q+1}$ decomposes into $\varphi = \varphi_1 + \varphi_2$. For given $v \in \Delta_{p+1,q+1}$ we let $\varphi_v(x) := x \cdot v$ for $x \in \mathbb{R}^{p+1,q+1}$, yielding a twistor spinor on $\mathbb{R}^{p+1,q+1}$. Using the relation between the spinor derivatives $\nabla^{\mathbb{R}^{p+1,q+1}}$ and $\nabla^{\widehat{Q}^{p,q}}$ which is proved in [Lei01], namely

$$\nabla_X^{\mathbb{R}^{p+1,q+1}}\varphi_v|_{\widehat{Q}^{p,q}} = \nabla_X^{\widehat{Q}^{p,q}}\varphi_{v,1} + \left[\frac{1}{2}X\cdot\zeta_0\cdot\varphi_v\right]|_{\widehat{Q}^{p,q}} \text{ for } X \in T\widehat{Q}^{p,q} \subset T\mathbb{R}^{p+1,q+1}, \tag{4.3}$$

and using (2.9), one calculates that the induced spinor $\varphi_{v,1} \in \Gamma(Ann\left(\zeta_0+\zeta_{n+1}\right)) \cong \Gamma(S^{\widehat{Q}^{p,q},g_{St}})$ is a twistor spinor on $\left(\widehat{Q}^{p,q},g_{St}\right)$ with $\varphi_{v,2} \equiv 0$ for any $v \in \Delta_{p+1,q+1}$. As the map

$$\Delta_{p+1,q+1} \ni v \mapsto \varphi_v \mapsto \varphi_{v,1} \in \ker\ P^{g_{St}}\left(\widehat{Q}^{p,q}\right) \tag{4.4}$$

is clearly injective, we have for dimensional reasons that all twistor spinors on the homogeneous model arise this way. This enables us to give a global description of the zero set structure for twistor spinors on $\widehat{Q}^{p,q}$:

Proposition 4.5 *Let φ be a nontrivial twistor spinor on $(\widehat{Q}^{p,q}, g = g_{St})$. Suppose that there is $x \in Z_\varphi$. Then it holds that*

$$Z_\varphi = exp_x\left(ker\ D^g\varphi(x)\right) \ \ or\ Z_\varphi = \{x,-x\}. \tag{4.5}$$

Remark 4.6 Note that Proposition 7.24 generalizes a classical result from [Lic89] for the Riemannian case: A twistor spinor on the standard sphere admits at most one zero. This follows from Proposition 4.5 as in the Riemannian case $Q^{0,q} = \{-1,1\} \times S^q$ and by (4.5) every twistor spinor on $\{-1,1\} \times S^q$ with zero has zero set a point or $\{(+1,x),(-1,-x)\}$ which is also a point if intersected with one of the spheres.

Proof. We find a unique nontrivial $v \in \Delta_{p+1,q+1}$ such that $\varphi = \varphi_{v,1}$ is induced by a twistor spinor φ_v on $\mathbb{R}^{p+1,q+1}$ by means of the map (4.4). We calculate $D^g\varphi$. To this end, let $f \in T\widehat{Q}^{p,q}$ be a unit-length vector. Then (4.3) yields that

$$g(f,f)\cdot f \cdot \nabla_f^{\mathbb{R}^{p+1,q+1}} \varphi_v|_{\widehat{Q}^{p,q}} = g(f,f)\cdot f\cdot \nabla_f^{\widehat{Q}^{p,q}}\varphi_{v,1} - \frac{1}{2}\cdot\zeta_0\cdot\varphi_v|_{\widehat{Q}^{p,q}}.$$

Imposing the twistor equations for φ_v and $\varphi_{v,1}$ then leads to

$$\frac{1}{n+2}\cdot D^{\langle\cdot,\cdot\rangle_{p+1,q+1}}\varphi_v|_{\widehat{Q}^{p,q}} = \frac{1}{n}D^g\varphi_{v,1} - \frac{1}{2}\cdot\zeta_0\cdot\varphi_v|_{\widehat{Q}^{p,q}}.$$

However, on $\mathbb{R}^{p+1,q+1}$ we have that $D^{\langle\cdot,\cdot\rangle_{p+1,q+1}}\varphi_v = -(n+2)\cdot v$, and this implies $D^g\varphi_{v,1}(y) = n\cdot(-v + \frac{1}{2}\zeta_0\cdot y\cdot v)$ for all $y \in \widehat{Q}^{p,q}$.
In particular, $x \in Z_{\varphi_{v,1}} = Z_{\varphi_v} \cap \widehat{Q}^{p,q} = \{x \in \widehat{Q}^{p,q} \mid x\cdot v = 0\}$ yields that

$$\ker D^g\varphi(x) = \{t \in T_x\widehat{Q}^{p,q} \mid t\cdot v = 0\}. \tag{4.6}$$

Let the zero x be fixed and fix some $b = (b_1,b_2) \in T_x\widehat{Q}^{p,q}\backslash\{0\}$ with $g_x(b,b) = \langle b,b\rangle_{p+1,q+1} = 0$. One checks that the geodesic through x in direction b is given by $\delta_b(t) = \cos(t||b_1||)\cdot x + \sin(t||b_1||)\cdot\frac{b}{||b_1||}$ with $||\cdot||$ being the standard Euclidean norm on $\mathbb{R}^{p+1}$, as $\delta_b''(t) = -||b_1||^2_{p+1}\delta_b(t)$. If now additionally $b\cdot v = 0$, we have that $\delta_b(1)\cdot v = 0$ as $x \in Z_\varphi$, i.e. $x\cdot v = 0$. This shows the $\supset$-direction in (4.5).
On the other hand, suppose that $y = (y_1,y_2) \in Z_\varphi$. As $y\cdot v = x\cdot v = 0$, it follows that $0 = (y\cdot x + x\cdot y)\cdot v = -2\langle x,y\rangle_{p+1,q+1}v$, i.e. $\langle x_1,y_1\rangle_{p+1} = \langle x_2,y_2\rangle_{q+1}$. Since $\langle x_i,x_i\rangle = \langle y_i,y_i\rangle = 1$ for $i = 1,2$, we find $\alpha_i \in [0;\pi]$ and $d_1 \in \mathbb{R}^{p+1}, d_2 \in \mathbb{R}^{q+1}$ with $\langle x_i,d_i\rangle = 0$ and $||d_1|| = ||d_2|| = 1$ such that $y_i = \cos(\alpha_i)\cdot x_i + \sin(\alpha_i)\cdot d_i$ for $i = 1,2$. The condition $\langle x_1,y_1\rangle = \langle x_2,y_2\rangle$ then leads to $\alpha_1 = \alpha_2 = \alpha$. Thus,

$$y = \cos(\alpha)\cdot x + \sin(\alpha)\cdot d$$

for $d = (d_1,d_2) \in T_x\widehat{Q}^{p,q}$. If $\sin(\alpha) \neq 0$, we conclude that $d\cdot v = 0$, and thus by (4.6) we have $d \in \ker D^g\varphi(x)$. As moreover $||d_1|| = 1$, we see that there is $t \in \mathbb{R}$ with $y = \cos(t||d_1||)\cdot x + \sin(t||d_1||)\cdot\frac{d}{||d_1||} = \delta_d(t) = \delta_{td}(1)$, where δ_d is the maximal geodesic through x in direction of d. Consequently, $y \in \exp_x(\ker D^g\varphi(x))$ for this case. If $\sin(\alpha) = 0$, we have either that $y = x$ where the statement is trivial or $y = -x$. If $\ker D^g\varphi(x)$ is nontrivial in this situation, we may choose arbitrary $d \in \ker D^g\varphi(x)\backslash\{0\}$ for a geodesic δ_d joining x and $-x$. Otherwise $\ker D^g\varphi(x) = 0$ and the situation $Z_\varphi = \{x,-x\}$ occurs. □

Remark 4.7 The above proof further yields the following for the flat model: In the notation of the proof of Proposition 4.5, let $x \in Z_\varphi$ and suppose that for some $w = (w_1,w_2) \in T_x\widehat{Q}^{p,q} \cap W$, where $W := \{w \in T_x\widehat{Q}^{p,q} \mid ||w_1|| + ||w_2|| < \frac{\pi}{2}\} \subset \mathbb{R}^{p+1,q+1}$ it holds that $y := \exp_x(w) = \delta_w(1) \in Z_\varphi$. As $\langle y,y\rangle_{p+1,q+1} = 0$ it follows that $w_1 \neq 0$ and $w_2 \neq 0$ or $y = x$.

Ignoring the last case, we see that the maximal geodesic δ_w through x in direction w is thus given by

$$\delta_w(t) = \cos(t||w_1||)\cdot x_1 + \sin(t||w_1||)\cdot \frac{w_1}{||w_1||} + \cos(t||w_2||)\cdot x_2 + \sin(t||w_2||)\cdot \frac{w_2}{||w_2||}.$$

Now $x, y \in Z_\varphi$ implies that $\langle x, \delta_w(1)\rangle_{p+1,q+1} = 0$ which yields that $\cos^2(||w_1||) = \cos^2(||w_2||)$. However, $w \in W$ then leads to $||w_1|| = ||w_2||$. Consequently, $\langle w, w\rangle_{p+1,q+1} = 0$. Now $y\cdot v = x\cdot v = 0$ and the form of $\delta_w(t)$ leads to $w\cdot v = 0$ as in the proof of the previous Proposition. This shows that $w \in \ker D^g\varphi(x)$.
Therefore, if $x \in Z_\varphi$, this observation together with Proposition 7.24 shows that for every open neighbourhood V of 0 in $T_x\widehat{Q}^{p,q}$ sufficiently small such that $V \subset W$ and setting $N' := \exp_x(V)$ we have on the homogeneous model

$$Z_\varphi \cap N' = \exp_x\left(\ker D^g\varphi(x) \cap V\right). \tag{4.7}$$

4.2 Proof of general result

We return to general twistor spinors on arbitrary conformal spin manifolds $(M^{p,q}, c)$. The main idea for the proof of Theorem 4.3 is that locally the description of Z_φ can always be reduced to the homogeneous model $\widehat{Q}^{p,q}$ which we have already studied. The technical tool which makes this relation precise is the holonomy reduction procedure for general Cartan geometries as introduced in [CGH14]. Applied to our setting, this reads as follows:

Let $\psi \in \Gamma(\mathcal{S}(M))$ be a ∇^{nc}-parallel spin tractor. We view $\mathcal{S}(M) = \overline{\mathcal{Q}}^1_+ \times_{\widetilde{B}^+} \Delta_{p+1,q+1}$, where $\widetilde{B}^+ = Spin^+(p+1, q+1)$. By standard principle bundle theory, ψ then corresponds to a $\widetilde{B}^+$–equivariant smooth map $\widehat{\psi} : \overline{\mathcal{Q}^1_+} \to \Delta_{p+1,q+1}$, where $u \mapsto [u]^{-1}\psi(\pi(u))$. As ψ is parallel, the image $\mathcal{O} := \widehat{\psi}\left(\overline{\mathcal{Q}^1_+}\right) \subset \Delta_{p+1,q+1}$ constitutes a single orbit wrt. the $\widetilde{B}^+$-action on $\Delta_{p+1,q+1}$, called the $\widetilde{B}^+$-type of ψ. We now bring into play that ∇^{nc} is induced by $\left(\mathcal{Q}^1_+, \widetilde{\omega}^{nc}\right)$, being a Cartan geometry of type $(\widetilde{B}^+, \widetilde{P}^+ = \lambda^{-1}(Stab_{\mathbb{R}^+e_-}SO^+(p+1, q+1)))$ where naturally $\mathcal{Q}^1_+ \subset \overline{\mathcal{Q}}^1_+$. For $x \in M$ we define the $\widetilde{P}^+$–type of x wrt. ψ to be the $\widetilde{P}^+$–orbit $\widehat{\psi}\left(\left(\mathcal{Q}^1_+\right)_x\right) \subset \mathcal{O} \subset \Delta_{p+1,q+1}$ which may change over x. M then decomposes into a disjoint union according to $\widetilde{P}^+$–types, each of which is an initial submanifold of M as studied in detail in [CGH14] for general Cartan geometries. In this special Cartan setting, Proposition 2.7 from [CGH14] now goes as follows:

Proposition 4.8 *Let $(M^{p,q}, c)$ be a conformal spin manifold and let $\psi \in \Gamma(\mathcal{S}(M))$ be a parallel spin tractor on $\left(\mathcal{Q}^1_+ \to M, \widetilde{\omega}^{nc}\right)$. For given $g \in c$ denote by $\varphi = \widetilde{\Phi}^g(proj^g_+\psi) \in \Gamma(S^g)$ the corresponding twistor spinor. Let $x \in Z_\varphi \subset M$. Then there is a parallel spin tractor $\phi \in \Gamma\left(\mathcal{S}\left(\widehat{Q}^{p,q}\right)\right)$ on the homogeneous model $\left(\widetilde{B}^+ \to \widetilde{B}^+/\widetilde{P}^+ = \widehat{Q}^{p,q}, \omega^{MC}\right)$ for which $x' := e\widetilde{P}^+ \in \widetilde{B}^+/\widetilde{P}^+$ has the same $\widetilde{P}^+$–type wrt. ϕ that x has wrt. ψ. Further, let $\varphi' := \widetilde{\Phi}^{g_{St}}(proj^{g_{St}}_+\phi)$ be the associated twistor spinor wrt. the conformally flat metric g_{St} on $\widehat{C}^{p,q} = (\widehat{Q}^{p,q}, [g_{St}])$. Then there are open neighbourhoods N of x in M and N' of x' in $\widehat{Q}^{p,q}$ and a diffeomorphism $\Theta : N \to N'$ such that $\Theta(x) = x'$ and*

$$\Theta\left(Z_\varphi \cap N\right) = Z_{\varphi'} \cap N'. \tag{4.8}$$

As a direct consequence of (4.7) we may in the notation of Proposition 4.8 after eventually restricting N and N' assume that

$$\Theta(Z_\varphi \cap N) = Z_{\varphi'} \cap N' = \exp_{x'}\left(\ker\, D^{g_{St}}\varphi'(x') \cap V\right), \tag{4.9}$$

where V is as in (4.7). The proof of Theorem 4.3 further needs the following very technical result:

Lemma 4.9 *In the notation of Proposition 4.8, it holds for the zero $x \in Z_\varphi$ that*

$$\dim\, \ker\, D^g\varphi(x) = \dim\, \ker\, D^{g_{St}}(x'). \tag{4.10}$$

Proof. For the proof of (4.10), we note that in the notation of Proposition 4.8 it holds

$$\begin{aligned} \psi(x) = [[\widetilde{\sigma}^g(l), e], e_- w] &\overset{\text{Thm.3.3}}{\Rightarrow} -\frac{1}{n}D^g\varphi(x) = [l, \chi(e_- w)], \\ \phi(x') = \left[[\widetilde{\sigma}^{g_{St}}(l'), e], e_- w'\right] &\Rightarrow -\frac{1}{n}D^{g_{St}}\varphi'(x') = [l', \chi(e_- w')] \end{aligned} \tag{4.11}$$

for spinors $w, w' \in \Delta_{p+1,q+1}$ and $l \in \mathcal{Q}^g_x(M)$, $l' \in \mathcal{Q}^{g_{St}}_{x'}(\widehat{Q}^{p,q})$. As the $\widetilde{P}^+$–types of $\psi(x)$ and $\phi(x')$ coincide by Proposition 4.8, there is by definition $\widetilde{p} \in \widetilde{P}^+$ such that

$$\widetilde{p} \cdot (e_- w) = e_- \cdot w'. \tag{4.12}$$

We therefore investigate the $\widetilde{P}$-action on $Ann(e_-) \subset \Delta_{p+1,q+1}$ more closely. Consider the 2-fold covering $\lambda : Spin(p+1, q+1) \to SO(p+1, q+1)$ which is explicitly given by $\lambda(u)(x) = u \cdot x \cdot u^{-1}$ (cf. [Bau81]), i.e.

$$\widetilde{p} \cdot x = \lambda(\widetilde{p})(x) \cdot \widetilde{p}. \tag{4.13}$$

By (2.15) there are $a \in \mathbb{R}^+$, $A \in SO(^+p, q)$ and $v \in (\mathbb{R})^*$ such that wrt. the splitting $\mathbb{R}^{p+1,q+1} \cong \mathbb{R}e_- \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}e_+$ we have that $\lambda(\widetilde{p}) = \begin{pmatrix} a^{-1} & v & -\frac{1}{2}a\langle v, v\rangle_{p,q} \\ 0 & A & -aAv^\sharp \\ 0 & 0 & a \end{pmatrix}$. By means of spinor representation, we view the element $\widetilde{p} \in Spin^+(p+1, q+1)$ as $\widetilde{p} : \Delta_{p+1,q+1} \to \Delta_{p+1,q+1}$, i.e. as a linear isomorphism acting on spinors. Conjugating with the isomorphism Π from (1.11) this can be equivalently described as $\widetilde{p}_\Pi := \Pi \circ \widetilde{p} \circ (\Pi)^{-1} : \Delta_{p,q} \oplus \Delta_{p,q} \to \Delta_{p,q} \oplus \Delta_{p,q}$, which in matrix form is given by

$$\widetilde{p}_\Pi = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \text{ for } X, Y, Z, W : \Delta_{p,q} \to \Delta_{p,q}.$$

Clearly, under this identification, (4.13) is equivalent to $\widetilde{p}_\Pi \circ cl(x) = cl(\lambda(\widetilde{p})(x)) \circ \widetilde{p}_\Pi$, where

$$cl(x) : \underbrace{\Delta_{p,q} \oplus \Delta_{p,q}}_{=\Pi(\Delta_{p+1,q+1})} \to \underbrace{\Delta_{p,q} \oplus \Delta_{p,q}}_{=\Pi(\Delta_{p+1,q+1})} \tag{4.14}$$

is Clifford multiplication with $x \in \mathbb{R}^{p+1,q+1}$.
Let $x = e_-$. As for any $w \in \Delta_{p+1,q+1}$ we have that $e_-(e_- w + e_+ w) = e_- e_+ w$, we see by (1.11) that $cl(e_-) = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ and $cl(\lambda(\widetilde{p})(e_-)) = a^{-1} cl(e_-)$. Then (4.13) yields that

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

which implies that $Y = 0$ and $W = a^{-1}X$. Now we let $x \in \mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q+1}$. Note that we have to carefully distinguish between the Clifford action of x on $\Delta_{p+1,q+1}$ in (4.14) and the Clifford action $cl_{p,q}(x) : \Delta_{p,q} \to \Delta_{p,q}$ if x is considered as element of $\mathbb{R}^{p,q}$. As for all $w \in \Delta_{p+1,q+1}$ the equality $\Pi(x \cdot (e_-w + e_+w)) = \begin{pmatrix} -x \cdot \chi(e_-e_+w) \\ x \cdot \chi(e_-w) \end{pmatrix}$ holds, we have that $cl(x) = \begin{pmatrix} -cl_{p,q}(x) & 0 \\ 0 & cl_{p,q}(x) \end{pmatrix}$. Moreover, $\lambda(\widetilde{p})(x) = \begin{pmatrix} \langle v, x\rangle_{p,q} \\ Ax \\ 0 \end{pmatrix}$, i.e. $cl(\lambda(\widetilde{p})(x)) = \begin{pmatrix} -cl_{p,q}(Ax) & 0 \\ \langle v, x\rangle_{p,q} & cl_{p,q}(Ax) \end{pmatrix}$. Now (4.13) yields that

$$\begin{pmatrix} X & 0 \\ Z & a^{-1}X \end{pmatrix} \cdot \begin{pmatrix} -cl_{p,q}(x) & 0 \\ 0 & cl_{p,q}(x) \end{pmatrix} = \begin{pmatrix} -cl_{p,q}(Ax) & 0 \\ \langle v, x\rangle_{p,q} & cl_{p,q}(Ax) \end{pmatrix} \cdot \begin{pmatrix} X & 0 \\ Z & a^{-1}X \end{pmatrix}$$

which can easily be shown to be equivalent to

$$cl_{p,q}(Ax) \cdot X = X \cdot cl_{p,q}(x) \ \forall x \in \mathbb{R}^{p,q}. \tag{4.15}$$

The condition (4.12) is under our identifications clearly equivalent to

$$\widetilde{p}_{\Pi} \begin{pmatrix} 0 \\ \chi(e_-w) \end{pmatrix} = \begin{pmatrix} X & 0 \\ Z & a^{-1}X \end{pmatrix} \begin{pmatrix} 0 \\ \chi(e_-w) \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ \chi(e_- \cdot w') \end{pmatrix}$$

from which follows that $a^{-1}X(\chi(e_-w)) = \chi(e_- \cdot w')$. This leads to the following implications for $x, y \in \mathbb{R}^{p,q}$:

$$x \cdot \chi(e_-w) = 0 \Rightarrow 0 = X(x \cdot \chi(e_-w)) \overset{(4.15)}{=} (Ax) \cdot (X(\chi(e_-w))) = a(Ax) \cdot \chi(e_- \cdot w'),$$

$$(Ay) \cdot \chi(e_- \cdot w') = 0 \overset{(4.15)}{\Rightarrow} X(y \cdot \chi(e_-w)) = 0 \Rightarrow y \cdot \chi(e_-w) = 0,$$

where in the last step we used that $X \in GL(\Delta_{p,q})$ as $\widetilde{p} \in GL(\Delta_{p+1,q+1})$. Thus we have by the last two lines that $A(\ker\ \chi(e_-w)) = \ker\ \chi(e_-w')$ which by (4.11) proves the Proposition. □

As a last ingredient for the Proof of Theorem 4.3 we need the following Lemma whose proof can be found in [Lei01] and which is of interest in its own right:

Lemma 4.10 *Let $\varphi \in ker\ P^g$ be a twistor spinor on a pseudo-Riemannian spin manifold $(M^{p,q}, g)$ with zero $x \in Z_\varphi$. Let $v \in T_xM$ and let γ_v denote the maximal geodesic starting in x in direction v.*

1. *If $v \in ker\ D^g\varphi(x)$, then also $Im(\gamma) \subset Z_\varphi$.*
2. *If $v \notin ker\ D^g\varphi(x)$, then there is a neighbourhood $U = U(x,v)$ of x in M such that $Z_\varphi \cap U \cap Im(\gamma) = \{x\}$.*

Proof of Theorem 4.3: We first show that the quantity dim ker $D^g\varphi(x)$ does not depend on the zero $x \in Z_\varphi$ [1]. One way to see this is the structure of the parallel tractor form α_ψ^{p+1}. As shown in [Lei09] it holds for every $x \in Z_\varphi$ that

$$\alpha_\psi^{p+1}(x) \overset{g}{=} d \cdot s_-^\flat(x) \wedge \alpha^p_{D^g\varphi}(x) \text{ for } x \in Z_\varphi.$$

[1] Moreover, it does not depend on the chosen metric in the conformal class as can be seen directly from the transformation formulas in Lemma 2.35

for some nonzero constant $d \in \mathbb{R}$. Applying Lemma 1.24 to this then yields that

$$\dim \ker D^g\varphi(x) = \dim \ker \psi(x) - 1,$$

and the right side of this equation does not depend on $x \in Z_\varphi$ as ψ is parallel.
We next show that the zero set Z_φ is an embedded submanifold of M. To this end, let $x \in Z_\varphi$ be arbitrary. In the setting and notation of Proposition 4.8 and (4.9) we choose neighbourhoods N and N' where we may assume that $N' = \exp_{x'}(V)$ is a normal neighbourhood of x' as in (4.9) and we then consider $\widetilde{\Theta} := (\exp_{x'})^{-1}_{|V} \circ \Theta : N \to V$. Propositions 4.8 and (4.9) yield that $\widetilde{\Theta}(Z_\varphi \cap N) = \ker D^g\varphi'(x') \cap V$. We may compose this map with a linear isomorphism $A_{x'} : T_{x'}\widehat{Q}^{p,q} \to \mathbb{R}^n$ satisfying $A_{x'}(\ker D^g\varphi'(x')) = \mathbb{R}^k \times \{0\}$, where k does not depend on the choice of zero as seen above and in this way we obtain a submanifold chart for Z_φ around x. By Lemma 4.9 the dimension of this submanifold is $k = \dim \ker D^{g_{St}}\varphi'(x') = \dim \ker D^g\varphi(x)$. Moreover, this submanifold is totally lightlike, since for every curve γ in M with $\mathrm{Im}\,\gamma \subset Z_\varphi$ the twistor equation yields that $\gamma'(t) \cdot D^g\varphi(\gamma(t)) = 0$ from which $\gamma'(t) \in \ker D^g\varphi(\gamma(t))$ and $g(\gamma'(t), \gamma'(t)) = 0$ follows as $D^g\varphi(\gamma(t)) \neq 0$. It is moreover totally geodesic: By the above discussion we clearly have for $x \in Z_\varphi$ that $T_xZ_\varphi = \ker D^g\varphi(x) \subset T_xM$, and thus, if γ is a geodesic starting in x in direction $\ker D^g\varphi(x)$, Lemma 4.10 yields that the image of the geodesic is contained in Z_φ. It remains to prove the local formula (4.1). Lemma 4.10 yields that for every $x \in Z_\varphi$ one has that $\exp_x(\ker D^g\varphi(x) \cap D_x) \subset Z_\varphi$, where D_x is the maximal starshaped domain of definition for the exponential map at x. We restrict D_x to a sufficiently small open neighbourhood $\widetilde{D}_x$ of $0 \in T_xM$ such that $\exp_x\left(\ker D^g\varphi(x) \cap \widetilde{D}_x\right) \subset M$ is a k-dimensional submanifold. Consequently, it holds that

$$\exp_x\left(\ker D^g\varphi(x) \cap \widetilde{D}_x\right) \subset Z_\varphi \subset M,$$

where $Z_\varphi \subset M$ is as we have already seen also a k-dimensional submanifold from which we conclude that $\exp_x\left(\ker D^g\varphi(x) \cap \widetilde{D}_x\right)$ is an open submanifold of the embedded submanifold Z_φ. This yields (4.1) for arbitrary dimensions. □

Remark 4.11 The shape of the zero set of a twistor spinor on a Lorentzian manifold has already been studied in [Lei01], however with a different method: In the Lorentzian case one has that $Z_\varphi = Z_{V_\varphi}$, and thus one gets the shape of Z_φ by studying the zero set structure of conformal vector fields with certain additional properties. Thus, our tractor-method based proof can be viewed as an independent proof of this result.

Remark 4.12 As a direct consequence of Theorem 4.3, the connected components of the zero set of a nontrivial twistor spinor consist either of an isolated point or of the image of a null-geodesic or of a totally null-plane etc. A mixture of two of these geometric objects cannot occur as the zero set of one single twistor spinor, i.e. the *connected components of its zero set have the same dimension.* The whole local geometry of the zero set is encoded in the quantity $\dim \ker D^g\varphi(x)$ where $x \in Z_\varphi$ is an arbitrary zero. This has interesting *global* consequences. For instance, if we know a twistor spinor φ only on a small open set $U \subset M$ and we find that φ admits an isolated zero on U, then all zeroes of φ in M are isolated! In case of a Ricci-parallel metric in the conformal class one has stronger results about the shape of the set V appearing in (4.1) as explained in detail in [Lei01].

4.3 Projective structures on the zero set

We next discuss the geometric structure on the zero set submanifold $Z_\varphi \subset M$ in more detail. Our main result is that the conformal class naturally induces a projective structure on the zero set of a twistor spinor. Recall that two linear connections ∇ and $\widehat{\nabla}$ on a manifold N are called projectively equivalent iff there exists a 1-form $\Upsilon \in \Omega^1(N)$ such that

$$\widehat{\nabla}_X Y = \nabla_X Y + \Upsilon(Y)X + \Upsilon(X)Y \ \forall X, Y \in \mathfrak{X}(N).$$

Clearly, ∇ and $\widehat{\nabla}$ have the same torsion. A more geometric interpretation (cf. [CS09]) is that two linear connections with the same torsion are projectively equivalent if and only if they admit the same geodesics as unparametrized curves. A projective structure on N is an equivalence class of connections.

Proposition 4.13 *Let $\varphi \in \Gamma(S^g)$ be a nontrivial twistor spinor with $Z_\varphi \neq \varnothing$ on (M, c). Then for every $g \in c$ the Levi-Civita connection ∇^g descends to a torsion-free linear connection ∇ (as to be defined in (4.16)) on Z_φ. If g and $\widetilde{g}$ are conformally equivalent, the induced connections ∇ and $\widetilde{\nabla}$ are projectively equivalent, i.e., there is a natural construction*

$$\varphi \text{ on } (M, c) \to (Z_\varphi, [\nabla])$$

from conformal structures admitting a twistor spinor with zero to torsion-free projective structures on the zero set.

Proof. It follows directly from (4.1) that for $x \in Z_\varphi$ the tangent space to the zero set is given by $T_x Z_\varphi = \ker D^g\varphi(x) \subset T_x M$. In particular, $\ker D^g\varphi(x)$ does not depend on the choice of $g \in c$. For given $X, Y \in \mathfrak{X}(Z_\varphi)$ and $x \in Z_\varphi$ let $\gamma : (-\epsilon, \epsilon) \to M$ denote the maximal geodesic in M with $\gamma(0) = x$ and $\gamma'(0) = X(x)$. By Lemma (4.10) we have that $Im(\gamma) \subset Z_\varphi$. Consequently, we may consider $Y \circ \gamma$ and set

$$\nabla_X Y(x) := \left(\frac{\nabla^g}{dt}(Y \circ \gamma) \right)(0), \tag{4.16}$$

where $\frac{\nabla^g}{dt}$ denotes the induced derivative along γ. We have to check that $\nabla_X Y \in \mathfrak{X}(Z_\varphi)$. As $(Y \cdot D^g\varphi) \circ \gamma = 0$, it follows that

$$0 = \frac{\nabla^{S^g}}{dt}((Y \cdot D^g\varphi) \circ \gamma) = ((\nabla_X Y) \cdot D^g\varphi) \circ \gamma + (Y \circ \gamma) \cdot \underbrace{\frac{\nabla^{S^g}}{dt}(D^g\varphi \circ \gamma)}_{= \frac{n}{2} K^g(X \circ \gamma) \cdot (\varphi \circ \gamma) = 0}.$$

Consequently, $\nabla_X Y(x) \in \ker D^g\varphi(x) = T_x Z_\varphi$. Clearly, this holds for every metric in the conformal class. The fact that ∇ is torsion-free follows directly from the corresponding property of ∇^g. Now let $\widetilde{g} = e^{2\sigma} g$ be a conformally equivalent metric with associated connection $\widetilde{\nabla}$ on Z_φ. There is the well-known transformation formula (cf. [Fis13])

$$\nabla^{\widetilde{g}}_X Y = \nabla^g_X Y + X(\sigma)Y + Y(\sigma)X - g(X, Y) \cdot \mathrm{grad}^g \sigma.$$

As for $x \in Z_\varphi$ the space $\ker D^g\varphi(x)$ is totally lightlike, it is a direct consequence of the definition of ∇ that for all $X, Y \in \mathfrak{X}(Z_\varphi)$ we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + d\widehat{\sigma}(X) \cdot Y + d\widehat{\sigma}(Y) \cdot X,$$

where $\widehat{\sigma} := \sigma_{|Z_\varphi}$. It follows that ∇ and $\widehat{\nabla}$ are projectively equivalent. □

Remark 4.14 Note that as a direct consequence of the definitions it holds that $i^*R^g = R^\nabla$, where $i : Z_\varphi \hookrightarrow M$ and R^∇ is the curvature tensor of the connection ∇. In particular, if c admits a flat representative then so does $[\nabla]$.
Moreover, as dim $Z_\varphi \leq \min(p,q)$, the induced projective structure on the zero set, which is trivial for a point or a curve as are the only possibilities in Riemannian or Lorentzian geometry, becomes more interesting with increasing index of $[g]$. For metrics of signature $(2,q)$, for instance, one can have 2-dimensional projective structures on Z_φ which are in general non-trivial. Consequently, this new relation between projective and conformal geometry motivates to study the twistor equation in arbitrary signatures.

The induced projective structure on the zero set as follows from Proposition 4.13 can also be expressed and understood in terms of Cartan geometries:

Projective Cartan geometry on the flat model
According to Remark 2.23, we identify $\widehat{Q}^{p,q} = \widetilde{B}^+/\widetilde{P}^+$, where $B^+ = SO^+(p+1,q+1)$, with the space of all null rays in $\mathbb{R}^{p+1,q+1}$ equipped with a natural conformal structure $[g_{St}]$. Let $\varphi \in \Gamma(\widehat{Q}^{p,q}, S^{g_{St}})$ be a twistor spinor with zero on $\widehat{Q}^{p,q}$. As seen in (4.4), φ arises as the restriction of a unique twistor spinor of the form $\mathbb{R}^{p+1,q+1} \ni x \mapsto x \cdot v$ for some $v \in \Delta_{p+1,q+1}$ and

$$Z_\varphi = \{t \in \widehat{Q}^{p,q} \mid t \cdot v = 0\} = (\ker\, v) \cap \widehat{Q}^{p,q}.$$

Let $k :=$ dim Z_φ. Clearly, dim ker $v = k+1$. We now consider the subgroup J of $B^+ = \lambda(\widetilde{B}^+)$, given as

$$J := \{j \in B^+ \mid j_{|Z_\varphi} = Z_\varphi\} = \{j \in B^+ \mid j_{|\ker\, v} = \ker\, v\},$$

i.e we restrict the group $B^+ \cong Conf^+(\mathcal{Q}^{p,q})$ of all orientation-preserving conformal diffeomorphisms of $\widehat{Q}^{p,q}$ to those which restrict to diffeomorphisms of Z_φ. We may without loss of generality assume that the ray $\mathbb{R}^+e_- \in \widehat{Q}^{p,q}$ lies in Z_φ, where $Stab_{\mathbb{R}^+e_-}B^+ = P^+$ [2]. Clearly, J acts transitively on Z_φ, whence

$$Z_\varphi \cong J/\mathrm{Stab}_{\mathbb{R}^+e_-}J.$$

J gives rise to the group

$$\widehat{J} := \{j_{|Z_\varphi} : Z_\varphi \to Z_\varphi \mid j \in J\}/Z(\{j_{|Z_\varphi} \mid j \in J\}),$$

i.e. we consider diffeomorphisms of the zero set which arise as restrictions of orientation-preserving conformal diffeomorphisms of the model space $\widehat{Q}^{p,q}$ and divide out the center. It is easy to verify that also $\widehat{J}$ acts transitively on Z_φ, i.e.

$$Z_\varphi \cong \widehat{J}/\mathrm{Stab}_{\mathbb{R}^+e_-}\widehat{J}.$$

By construction, also the Maurer Cartan form of B^+ restricts to that of J , i.e. the Cartan geometry $(\widetilde{B}^+ \to \widetilde{B}^+/\widetilde{P}^+ \cong \widehat{Q}^{p,q}, \omega^{MC}_{B^+})$ induces the Cartan geometry $(J \to J/\mathrm{Stab}_{\mathbb{R}^+e_-}J \cong Z_\varphi, \omega^{MC}_J)$, defined over the zero set. In order to obtain a better description of the homogeneous space $\widehat{J}/\mathrm{Stab}_{\mathbb{R}^+e_-}\widehat{J}$, we describe the involved groups in terms of matrices: Let

[2] If x_0 is any zero, i.e. $x_0 \cdot v = 0$, we find $g \in \widetilde{B}^+$ such that $\lambda(g)(x_0) = e_-$. It follows that $e_- \cdot (g \cdot v) = 0$. We can then work with the twistor spinor induced by $g \cdot v$ instead of v.

$(e_-, f_1, ..., f_k)$ be a (totally lightlike) basis of ker v. We can complete this to a basis for $\mathbb{R}^{p+1,q+1}$ such that wrt. this basis, the standard pseudo-Euclidean inner product $\langle \cdot, \cdot \rangle_{p+1,q+1}$ is given by $\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I_{p',q'} \end{pmatrix}$, where I is the $(k+1)\times(k+1)$ identity matrix, $p' = p-k, q' = q-k$ and $I_{p',q'}$ represents $\langle \cdot, \cdot \rangle_{p',q'}$ on $\mathbb{R}^{p'+q'}$. Wrt. this basis, J is in terms of matrices given by

$$J = \left\{ j = \begin{pmatrix} A & B & C \\ 0 & \left(A^T\right)^{-1} & 0 \\ 0 & F & G \end{pmatrix} \;\middle|\; \begin{matrix} A \in GL^+(k+1), B \in M(k+1), C \in M(k+1, p'+q') \\ F \in M(p'+q', k+1), G \in SO^+(p',q') \\ A^{-1}B + B^T(A^T)^{-1} + F^T I_{p',q'} F = 0, C^T\left(A^T\right)^{-1} + G^T I_{p',q'} F = 0 \end{matrix} \right\}.$$

J is a semidirect product $J = SL(k+1) \ltimes M$, where $SL(k+1)$ is embedded in $M(n+2)$ via $A \hookrightarrow \begin{pmatrix} A & 0 & 0 \\ 0 & \left(A^T\right)^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}$, and the group M is given as

$$M = \left\{ j_M = \begin{pmatrix} a \cdot I & V & X \\ 0 & a^{-1} \cdot I & 0 \\ 0 & F & G \end{pmatrix} \;\middle|\; \begin{matrix} a \in \mathbb{R}^+, V \in M(k+1), X \in M(k+1, p'+q') \\ F \in M(p'+q', k+1), G \in SO^+(p',q') \\ a^{-1}V + a^{-1}V^T + F^T I_{p',q'} F = 0, a^{-1}X^T + G^T I_{p',q'} F = 0 \end{matrix} \right\}.$$

Wrt. this decomposition, we write $j = (A, j_W)$ for $j \in J, A \in SL(k+1), j_M \in M$. It follows that $j \in \text{Stab}_{\mathbb{R}^+ e_-} J$ iff $A \in \text{Stab}_{\mathbb{R}^+ e_-} SL(k+1)$. Moreover, it follows immediately that

$$\widehat{J} \cong SL(k+1).$$

Factoring out the $M-$part of J then leads to a well-defined diffeomorphism

$$\begin{aligned} J/\text{Stab}_{\mathbb{R}^+ e_-} J &\overset{\cong}{\to} SL(k+1)/\text{Stab}_{\mathbb{R}^+ e_-} SL(k+1) \cong Z_\varphi, \\ (A, j_M) \cdot \text{Stab}_{\mathbb{R}^+ e_-} J &\mapsto A \cdot \text{Stab}_{\mathbb{R}^+ e_-} SL(k+1). \end{aligned}$$

Geometrically, omitting the $M-$part of J corresponds to restricting the elements of $J \subset Conf^+(\mathcal{Q}^{p,q})$ to the zero set Z_φ, yielding the group $\widehat{J}$. Consequently, we have constructed a naturally induced Cartan geometry

$$(SL(k+1) \to SL(k+1)/\text{Stab}_{\mathbb{R}^+ e_-} SL(k+1) \cong Z_\varphi, \omega^{MC}_{SL(k+1)})$$

over the zero set of φ. However, [CS09] shows that the homogeneous space $SL(k+1)/\text{Stab}_{\mathbb{R}^+ e_-} SL(k+1)$, considered as (parabolic) Cartan geometries of type $(SL(k+1), \text{Stab}_{\mathbb{R}^+ e_-} SL(k+1))$ is the homogeneous model for oriented projective structures. Clearly,

$$SL(k+1)/\text{Stab}_{\mathbb{R}^+ e_-} SL(k+1) \cong \{\text{rays in } \mathbb{R}^{k+1}\} \cong S^k,$$

and under this identification, the group $SL(k+1)$ is seen to be the group of all orientation-preserving diffeomorphisms of S^k which also preserve the projective structure, i.e. that map great circles into great circles.
Thus, there is a naturally induced Cartan geometry defined on the zero set of every twistor spinor of $\widehat{Q}^{p,q}$ which yields the flat model of projective geometry. The group $SO^+(p+1, q+1) \cong Conf^+(\widehat{Q}^{p,q})$ descends after restriction to Z_φ and dividing out the centre naturally to the group $\widehat{J} \cong SL(k+1)$ of orientation-preserving projective diffeomorphisms of Z_φ.

The curved case
Let us now briefly sketch the relation between a conformal structure admitting twistor spinors and induced projective Cartan geometries on their zero sets for the general setting: Every oriented and torsion-free $k-$dimensional projective structure $(N,[\nabla])$ can be equivalently described as Cartan geometry $(\mathcal{P}^{pr},\omega^{pr})$ of type $(SL(k+1),K)$, where $K = Stab_{\mathbb{R}^+e_1}SL(k+1)$ is the stabilizer of the ray spanned by the first basis vector $e_1 \in \mathbb{R}^{k+1}$ under the standard matrix action of $SL(k+1)$ on $\mathbb{R}^{k+1}$, and the Cartan connection $\omega^{pr} \in \Omega^1(\mathcal{P}^{pr},\mathfrak{sl}(k+1))$ is again distinguished by certain normalisation conditions, cf. [CS09] for details of the construction. As in the conformal case, the standard action of $SL(k+1)$ on $\mathbb{R}^{k+1}$ leads to an associated projective standard tractor bundle $\mathcal{T}^{pr}N = \mathcal{P}^{pr}\times_K\mathbb{R}^{k+1}$ with induced covariant derivative ∇^{pr}. Via some fixed $\nabla \in [\nabla]$ we have that (cf. [Arm06])

$$\mathcal{T}^{pr}N \overset{\nabla}{\cong} \underline{\mathbb{R}}\oplus TM, \text{ and } \nabla^{pr}_X\begin{pmatrix}\alpha\\Y\end{pmatrix} = \begin{pmatrix}X(\alpha)+P^{\nabla}(X,Y)\\ \nabla_XY+\alpha X\end{pmatrix}, \tag{4.17}$$

where $P^{\nabla} = \frac{1}{n-2}\mathrm{Ric}^{\nabla}$ (which in general is non symmetric).
Let us apply this to our original setting, i.e. consider an $n-$dimensional conformal spin manifold $(M,[g])$ with $\psi \in Par(\mathcal{S}(M),\nabla^{nc})$ and let $\varphi = \widetilde{\Phi}^g(proj^g_+(\psi)) \in \ker P^g$. Suppose that $k = \dim Z_\varphi > 0$. We restrict the standard tractor bundle $\mathcal{T}(M)\to M$ onto $\mathcal{T}(M)_{|Z_\varphi} \supset \ker \psi_{|Z_\varphi} \to Z_\varphi$, being a vector bundle of rank $k+1$ over Z_φ. For $g\in[g]$, the $g-$trivialization (2.23) yields that

$$\ker \psi_{|Z_\varphi} \overset{g}{\cong} \left\{ \begin{pmatrix}\alpha\\Y\\0\end{pmatrix} \mid \alpha\in\mathbb{R}, Y \in \ker D^g\varphi_{|Z_\varphi} \right\} \to Z_\varphi. \tag{4.18}$$

As $\ker D^g\varphi_{|Z_\varphi}$ is totally lightlike, it follows from (2.25) that the normal conformal Cartan connection ∇^{nc} restricts to a covariant derivative on the bundle $\ker \psi_{|Z_\varphi} \to Z_\varphi$. On the other hand, we can as in Proposition 4.13 form the projective structure $(Z_\varphi,[\nabla])$. Finally, (4.17), (4.18) and (2.25) show that there exists a natural isomorphism of vector bundles over Z_φ, namely

$$\boxed{\nu : \ker \psi_{|Z_\varphi} \to \mathcal{T}^{pr}Z_\varphi \text{ such that } \nu\circ\nabla^{nc} = \nabla^{pr}\circ\nu},$$

whose description wrt. some $g\in c$ and associated covariant derivative ∇ on Z_φ is explicitly given by $\nu\left(\begin{pmatrix}\alpha\\Y\\0\end{pmatrix}\right) = \begin{pmatrix}\alpha\\Y\end{pmatrix} \in \underline{\mathbb{R}}\oplus TZ_\varphi \overset{\nabla}{\cong} \mathcal{T}^{pr}Z_\varphi$.
That is, on the level of the underlying bundles the conformal Cartan Geometry $(\mathcal{Q}^1_+ \to M,\widetilde{\omega}^{nc})$ admitting a twistor spinor with zero restricts to the projective Cartan geometry $(\mathcal{P}^{pr}\to Z_\varphi,\omega^{pr})$ of type $(SL(k+1),K)$ defined over the zero set.

Proposition (4.13) motivates us to consider pseudo-Riemannian extensions which are treated in [HS11a] and [CGGV09]: Given any manifold N of dimension $k\geq 2$ equipped with a class of projectively equivalent torsion-free connections $[\nabla]$, there is a, in general non-natural, construction of a split-signature conformal structure $[g_\nabla]$ on T^*N. We elaborate on the very explicit approach to pseudo-Riemannian extensions developed in [CGGV09]: Let $x \in N$ and let $(x_1, ..., x_k)$ be local coordinates on $U \subset M$ around x. Fix ∇ in the projective class and let $\Gamma^{\nabla,t}_{ij}$ denote the local connection coefficients wrt. these coordinates. We introduce the Thomas projective parameters (cf. [Eas06]), defined by

$$\Pi^t_{ij} := \Gamma^{\nabla,t}_{ij} - \frac{1}{k+1}\Gamma^{\nabla,l}_{li}\delta^t_j - \frac{1}{k+1}\Gamma^{\nabla,l}_{lj}\delta^t_i$$

which are easily shown to be independent of $\nabla \in [\nabla]$. For the cotangent bundle $\pi : T^*N \to N$ let $(x_1, ..., x_k, x'_1, ..., x'_k)$ be cotangent coordinates on $\pi^{-1}(U)$ which result from writing $\omega \in \pi^{-1}(U)$ as $\omega = \sum_i x'_i dx^i$. We then define the pseudo-Riemannian extension of $(N, [\nabla])$ to be the split signature conformal structure on T^*N with representative

$$g_\nabla = 2dx^i \circ dx'^i - 2\sum_l x'_l \Pi^l_{ij} dx^i \circ dx^j.$$

[CGGV09] presents a coordinate-invariant definition of this metric, and in [HS11a] this conformal structure is shown to admit twistor spinors. Now given a pseudo-Riemannian conformal spin structure (M, c) admitting a twistor spinor φ with $k :=$dim $Z_\varphi > 2$, it is natural to ask how the following diagram could be completed:

$$\boxed{\begin{tikzcd}[column sep=small]
(M, c = [g]) \text{ with } \varphi \in \ker P^g \arrow[r, "\text{Prop. 4.13}"] \arrow[dr, dotted, "?"] & (Z_\varphi, [\nabla]) \text{ k-dim., torsion-free} \arrow[d, "\text{pseudo-Riem. extension}"] \\
& (T^*Z_\varphi, [g_\nabla]) \text{ of signature } (k,k)
\end{tikzcd}} \tag{4.19}$$

The best we could hope for is the existence of a conformal embedding of $(T^*Z_\varphi, [g_\nabla])$ into (M, c). A first step into this direction is given by the next Proposition:

Proposition 4.15 *In the setting of diagram (4.19) there exists for every $x_0 \in Z_\varphi$ an open neighbourhood V_{x_0} of $x_0 = (x_0, 0) \in T^*Z_\varphi$ and an embedding*

$$\Phi : V_{x_0} \to M \text{ such that } (\Phi^* g)_{(y,0)} = g_\nabla \ \forall (y, 0) \in Z_\varphi \subset V_{x_0}, \tag{4.20}$$

i.e. Φ is isometric along the zero section. Here, ∇ is the projective connection on Z_φ induced by ∇^g.

Proof. For fixed $g \in c$ and $x_0 \in Z_\varphi$, we consider normal coordinates $\widetilde{\beta} = (x_1, ..., x_n)$ on an open neighbourhood U_{x_0} of x_0 in M,

$$\widetilde{\beta} : U_{x_0} \overset{\exp^{-1}_{x_0|U_{x_0}}}{\to} T_{x_0}M \overset{A}{\to} \mathbb{R}^n,$$

which by Theorem 4.3 restrict to coordinates on $Z_\varphi \cap U_{x_0}$,

$$\widetilde{\beta} : Z_\varphi \cap U_{x_0} \to \ker\, D^g\varphi(x_0) \to \mathbb{R}^k.$$

Here, A is a linear isomorphism from $T_{x_0}M$ onto $\mathbb{R}^n$ whose restriction to ker $D^g\varphi(x_0)$ maps onto $\mathbb{R}^k \times \{0\}$. $\widetilde{\beta}$ induces cotangent coordinates $\beta = (x_1, ..., x_k, x'_1, ..., x'_k)$ on $T^*(Z_\varphi \cap U)$

in the usual way, i.e. for $(y,\omega) = \sum_i \omega_i dx^i_y$ we have that $\beta(y,\omega) = (\beta(y),\omega_1,...,\omega_k)$. By identifying Z_φ with the zero section in T^*Z_φ, we may then identify $\frac{\partial}{\partial x_i}_{|(y,\omega)} = \frac{\partial}{\partial x_i}_{|y} \in T_y Z_\varphi \subset T_{(y,\omega)}T^*Z_\varphi$ and $\frac{\partial}{\partial x'_i}_{|(y,\omega)} = dx^i_y \in T^*_y Z_\varphi$.

We find an open neighbourhood $V_{x_0} \subset T^*Z_\varphi$ of $x_0 \in T^*Z_\varphi$ such that

$$\Phi : V_{x_0} \to M,\ (y,\omega) \mapsto \exp_y(\omega^\sharp)$$

is well-defined. We compute the differential of Φ at $(x_0,0)$:

$$d\Phi_{(x_0,0)}\left(\frac{\partial}{\partial x_i}_{|(x_0,0)}\right) = \frac{d}{dt}_{|t=0}\exp_{\delta_i(t)}(0) = \frac{d}{dt}_{|t=0}\delta_i(t) = \frac{\partial}{\partial x_i}_{|x_0},$$
$$d\Phi_{(x_0,0)}\left(\frac{\partial}{\partial x'_i}_{|(x_0,0)}\right) = \frac{d}{dt}_{|t=0}\exp_{x_0}(\epsilon_i(t)^\sharp) = \left(dx^i_{x_0}\right)^\sharp,$$

where $\delta_i : I \to Z_\varphi \subset T^*Z_\varphi$ is a curve with $\delta_0 = x_0$ and $\frac{d}{dt}_{|t=0}\delta_i(t) = \frac{\partial}{\partial x_i}_{|x_0}$, $\epsilon : I \to T^*_y Z_\varphi \subset T^*Z_\varphi$ is a curve with $\epsilon(0) = x_0$ and $\frac{d}{dt}_{|t=0}\epsilon_i(t) = dx^i_y$.
The space $T_{x_0}Z_\varphi = \mathrm{span}\left(\frac{\partial}{\partial x_1}_{|x_0},...,\frac{\partial}{\partial x_k}_{|x_0}\right) \subset T_{x_0}M$ is totally lightlike wrt. g_{x_0}. We may choose a pseudo-orthonormal basis $(s_1,...,s_n)$ of $T_{x_0}M$ such that $\frac{\partial}{\partial x_i}_{|x_0} = s_i + s_{i+p}$. It follows that $\left(dx^i_{x_0}\right)^\sharp = -s_i + s_{i+p}$. This shows that the differential of Φ is injective, whence we obtain a local embedding after restricting V_{x_0} if necessary.

We now compute for $(y,0) \in V_{x_0}$:

$$\begin{aligned}
\Phi^*g_{(y,0)}\left(\frac{\partial}{\partial x_i}_{|(y,0)}, \frac{\partial}{\partial x_j}_{|(y,0)}\right) &= g_y\left(d\Phi_{(y,0)}\left(\frac{\partial}{\partial x_i}_{|(y,0)}\right), d\Phi_{(y,0)}\left(\frac{\partial}{\partial x_j}_{|(y,0)}\right)\right)\\
&= g_y\left(\frac{\partial}{\partial x_i}_{|y}, \frac{\partial}{\partial x_j}_{|y}\right) = 0 = (g_\nabla)_{(y,0)}\left(\frac{\partial}{\partial x_i}_{|(y,0)}, \frac{\partial}{\partial x_j}_{|(y,0)}\right),\\
\Phi^*g_{(y,0)}\left(\frac{\partial}{\partial x'_i}_{|(y,0)}, \frac{\partial}{\partial x'_j}_{|(y,0)}\right) &= g_y\left(\left(dx^i_{x_0}\right)^\sharp, \left(dx^j_{x_0}\right)^\sharp\right)\\
&= 0 = (g_\nabla)_{(y,0)}\left(\frac{\partial}{\partial x'_i}_{|(y,0)}, \frac{\partial}{\partial x'_j}_{|(y,0)}\right),\\
\Phi^*g_{(y,0)}\left(\frac{\partial}{\partial x_i}_{|(y,0)}, \frac{\partial}{\partial x'_j}_{|(y,0)}\right) &= g_y\left(\frac{\partial}{\partial x_i}_{|y}, \left(dx^j_{x_0}\right)^\sharp\right)\\
&= \delta_{ij} = (g_\nabla)_{(y,0)}\left(\frac{\partial}{\partial x'_i}_{|(y,0)}, \frac{\partial}{\partial x'_j}_{|(y,0)}\right).
\end{aligned}$$

By bilinear extension, this proves the Proposition. □

Remark 4.16 A related construction which characterizes anti-selfdual conformal structures in signature $(2,2)$ admitting distinguished spinor fields in terms of 2-dimensional projective structures is discussed in [DW07], for instance.

4.4 Geometry off the zero set

It is now natural to ask what can be said about the spinor and associated local geometries off the zero set if one knows the (local) structure of Z_φ. In the Riemannian case, a twistor spinor is always parallel on a Ricci-flat space off the zero set. For Lorentzian signature, F. Leitner showed that in case of an isolated zero the Lorentzian metric is locally off the zero set isometric to a static monopole $-dt^2+h$ where h is a Riemannian Ricci-flat metric with parallel spinor. If the zero is not isolated, then off the zero set the space is locally conformally equivalent to a Brinkmann space with parallel spinor.

Generalizing this, our results from section 4.3 reveal that in every signature the spinor is locally equivalent to a parallel spinor off the zero set. In fact, let $\psi \in \Gamma(\mathcal{S}(M))$ be a parallel spin tractor with associated twistor spinor $\varphi \in \Gamma(S^g)$ for $g \in c$. Let $x \in Z_\varphi$. It then holds *at* x that

$$\psi(x) = [[\widetilde{\sigma}^g(l(x)), e], 0 + e_- w] \tag{4.21}$$

for some $w \in \Delta_{p+1,q+1}$ and a local section $l : U \to \mathcal{Q}^g_+$. However, this means that $s_-(x) \in \ker \psi(x) \neq \{0\}$. In particular, since the dimension of this kernel is constant over M, Proposition 3.28 applies and yields the next statement.

Theorem 4.17 *Let $\varphi \in \Gamma(M, S^g)$ be a twistor spinor admitting a zero. Then there is an open dense subset $\widetilde{M} \subset M$ with $Z_\varphi \subset M \backslash \widetilde{M}$ such that for every $x \in \widetilde{M}$ there is an open neighbourhood $U_x \subset \widetilde{M}$ on which φ can be rescaled to a parallel spinor.*

Our discussion from section 3.3 implies further consequences relating the shape and dimension of the zero set to local geometric structures off the zero set. Let us for brevity assume that $[g]$ is a conformal class of metrics with $p \leq q$.

Proposition 4.18 *Let $\varphi \in \Gamma(S^g)$ be a twistor spinor with nonempty zero set Z_φ. Then there is a set of singular points $sing(\varphi) \subset M$ with $Z_\varphi \subset sing(\varphi)$ such that the following holds:*
There is $0 \leq k \leq p$ such that Z_φ is an embedded $k-$dimensional totally lightlike submanifold. On $M \backslash sing(\varphi)$, the spinor is locally conformally equivalent to a parallel spinor and the corresponding metric holonomy representation fixes a totally lightlike subspace of dimension k. If $k = p$ or $k = p-1$ there is even a fixed totally lightlike $k-$form. If $k = 0$, i.e. the zero is isolated, the spinor is off $sing(\varphi)$ locally parallel wrt. a Ricci-flat metric.

Proof. We observe first that for the number k appearing in the Proposition it holds that $k = \dim \ker D^g\varphi(x)$, where $x \in Z_\varphi$. Let $\psi \in \Gamma(M, \mathcal{S})$ be the parallel spin tractor associated to $\varphi \in \ker P^g$ and let $x \in Z_\varphi$ be arbitrary. Then $\psi(x)$ can be written as in (4.21), and in this notation we have that $-\frac{1}{n} \cdot D^g\varphi(x) = [l, \chi(e_- \cdot w)]$. Consequently,

$$\ker \psi(x) = \mathbb{R} \cdot s_-(x) \oplus \ker D^g\varphi(x) \subset \mathcal{I}_-(x) \oplus T_xM \oplus \mathcal{I}_+(x) \overset{g}{\cong} \mathcal{T}_x(M). \tag{4.22}$$

On the other hand, Proposition 3.28 and 3.34 apply: There exists an open, dense subset $\widetilde{M} \subset M$ such that on $\widetilde{M}$ there is locally a metric $g = g_U$ in the conformal class with $\dim \ker \psi_{|U} - 1 = \dim \ker \varphi_{|U}$. Comparing with (4.22) (note that $\dim \ker \psi$ is constant) yields that

$$\dim Z_\varphi = \dim \ker D^g \varphi_{|Z_\varphi} = k = \dim \ker \psi - 1 = \dim \ker \varphi_{|\widetilde{M}}. \tag{4.23}$$

Moreover, g can by Theorem 4.17 be chosen such that φ is locally parallel. In particular, $\ker \varphi$ is (locally) of the same dimension k as the zero set, parallel and totally lightlike (wrt. to a suitable metric in the conformal class) and henceforth fixed by the metric holonomy representation. The statement then directly follows from Proposition 3.34 which relates $\dim \ker \varphi$ to local geometries. □

Remark 4.19 The formula (4.23) derived in the previous proof shows that the zeroes of a twistor spinor know *global information* about the spinor in the sense that the dimension of the zero set controls the type of parallel spinor and geometry one has off the zero set. In particular, if the spinor admits an isolated zero at *some* point, then the metric can be locally rescaled to a Ricci-flat metric *everywhere* off a singular set.

As an application, let us consider the case $p = 2$ more carefully.

Remark 4.20 Let φ be a parallel spinor on $(M^{2,n-2}, g)$. The spinor leads to a nontrivial, parallel 2-form α^2_φ on M. The $SO^+(2, n-2)$-orbit type of this form must be one of the list from Remark 1.25: The first form, $\alpha^2_\varphi = l_1^\flat \wedge l_2^\flat$ corresponds to a parallel pure spinor. In the second case, $\alpha^2_\varphi = l^\flat \wedge t^\flat$, we can conclude that there is a nontrivial lightlike, parallel vector field and thus (U, g) is a Brinkmann space. In the third case, (U, g) is Ricci-flat (as $\ker \varphi = \{0\}$) and $Hol(M, g)$ leaves invariant a (possibly trivial) $n - 2m$ dimensional nondegenerate subspace $E^\perp$ and α^2_φ is Kähler on E. It follows with Remark (1.25) and standard holonomy theory that there is a local splitting $(U, g) \cong (U_1, g_1) \times (U_2, g_2)$, where the first factor is Ricci-flat pseudo Kähler of signature $(2, 2m-2)$ and the second factor (which might be trivial) is Riemannian Ricci-flat. Moreover both factors admit parallel spinors, since [Lei04] proves that a pseudo-Riemannian product admits a parallel spinor iff both factors admit a parallel spinors.

The discussion from Remark 4.20 together with the last statement directly leads to the following more concrete relation between the shape of the zero set and local geometries. All one has to observe is that $\dim Z_\varphi = \dim \ker \varphi$ off the zero set where φ is parallel.

Proposition 4.21 *Let $\varphi \in \Gamma(S^g)$ be a twistor spinor with zero on $(M^{2,n-2}, g)$. Then exactly one of the following cases occurs:*

1. *Z_φ consists locally of* totally lightlike planes. *In this case, the spinor is locally equivalent to a parallel spinor off the zero set and gives rise to a parallel totally lightlike 2-form.*

2. *Z_φ consists of* isolated images of lightlike geodesics. *In this case, the spinor is off the zero set locally conformally equivalent to a parallel spinor on a Brinkmann space.*

3. *Z_φ consists of* isolated points. *In this case there is for each point off the zero set an open neighbourhood and a local metric in the conformal class such that the resulting space is isometric to a product $(U_1, g_1) \times (U_2, g_2)$ where the first factor is Ricci-flat pseudo-Kähler and the second factor (which might be trivial) is Riemannian Ricci-flat. Both factors admit a parallel spinor.*

Finally, we study the behaviour of the Weyl tensor on the zero set of a twistor spinor:

Proposition 4.22 *Let $\varphi \in ker\ P^g$ be a twistor spinor with zero on (M,g) and suppose that dim $Z_\varphi \in \{0,1\}$. Then it holds that $W^g_{|Z_\varphi} = 0$.*

Proof. Let $x_0 \in Z_\varphi$. From the last integrability condition from Proposition 2.11 we obtain

$$(Z \lrcorner W^g_{x_0}(\eta)) \cdot \underbrace{D^g\varphi(x_0)}_{\neq 0} = 0 \ \forall Z \in T_{x_0}M, \eta \in \Omega^2(M). \tag{4.24}$$

In other words, we have that $W^g_{x_0}(X,Y,Z) \in \ker\ D^g\varphi(x_0)$ for all $X,Y,Z \in T_{x_0}M$. As dim Z_φ = dim ker $D^g\varphi(x_0)$ the claim follows if this kernel is trivial. Otherwise, we have that dim ker $D^g\varphi(x_0) = 1$ and proceed as follows: By (4.24) there exists a nonzero, isotropic vector $v \in T_{x_0}M$ such that $W^g_{x_0}(X,Y,Z) = \lambda(X,Y,Z) \cdot v$. We now fix a basis $(v,\widetilde{v},a_1,...,a_{n-2})$ of $T_{x_0}M$ where $\widetilde{v}$ is isotropic, $g_{x_0}(v,\widetilde{v}) = 1$ and $(a_1,...,a_{n-2})$ is a pseudo-orthonormal basis of $\{v,\widetilde{v}\}^\perp$. By definition, it holds that

$$W^g_{x_0}(X,Y,Z,U) = \lambda(X,Y,Z)g(v,U) = \begin{cases} 0 & U \in \text{span}(v,a_1,...,a_{n-2}) \\ \lambda(X,Y,Z) & U = \widetilde{v} \end{cases},$$

i.e. it remains to check that $W^g_{x_0}(X,Y,Z,\widetilde{v}) = 0$:

$$W^g_{x_0}(X,Y,Z,\widetilde{v}) = -W^g_{x_0}(X,Y,\widetilde{v},Z) = -\lambda(X,Y,\widetilde{v})g(v,Z)$$

The last expression, however, vanishes if $Z \perp v$ and in case $Z = \widetilde{v}$ it vanishes due to the symmetries of W^g. □

We do not know whether the previous result also holds in case of higher-dimensional zero sets.

Remark 4.23 Already in the Riemannian case the construction of non-conformally flat examples admitting twistor spinors with zeroes is quite involved, see [KR96, KR98]. We do not know whether there are any non-conformally flat examples of pseudo-Riemannian, non-Riemannian geometries admitting a twistor spinor with zero. [Lei07] presents a C^1-Lorentzian metric admitting a twistor spinor with isolated zero. It remains unclear whether this metric is also smooth at the zero.
Let us elaborate on the Lorentzian case and related problems in more detail. If φ is a twistor spinor with zero on a Lorentzian manifold, then V_φ is a essential conformal vector field, meaning that in cannot be rescaled to a Killing vector field for a metric in the conformal class. In the Lorentzian case we have $zero(V_\varphi) = zero(\varphi)$. Therefore, one has to search for Lorentzian geometries admitting conformal vector fields with zeroes which are not conformally flat around the zero set. It is easy to find examples of *spacelike* vector fields: In fact, consider on $\mathbb{R}^n$ with $n \geq 4$ and standard coordinates $(x^1,...,x^n)$ the Lorentzian metric g given by

$$g_{11} = g_{22} = g_{1j} = g_{2j} = 0, \ g_{12} = 1, \ g_{jk} = h_{jk}(t),$$

where $t = x^1$, the indices j,k,l,p,q always vary from 3 to n, and $t \mapsto h(t)$ is a smooth curve of symmetric positive-definite $(n-2) \times (n-2)$ matrices. The Christoffel symbols of g all vanish except possibly for Γ^2_{jk} and Γ^k_{1j}, characterized by $2\Gamma^2_{jk} = -\dot{g}_{jk}$ and $2\Gamma^k_{1j} = g^{kl}\dot{g}_{jl}$, so that the $(4,0)$ curvature components (or the Ricci-tensor components) are all zero except for those algebraically related to R_{1j1k}, with $4R_{1j1k} = -2\ddot{g}_{jk} + g^{pq}\dot{g}_{jp}\dot{g}_{kq}$ (or, respectively, except for $R_{11} = g^{jk}R_{1j1k}$). Thus, $W_{1j1k} = R_{1j1k} - (n-2)^{-1}R_{11}g_{jk}$ is in general nonzero. On the other hand, v given by

$$v^1 = 0,\ v^2 = 2x^2,\ v^j = x^j$$

is a g−conformal vector field, with the zero set consisting of the x^1 coordinate axis. Note that v is essential at each of its zeroes, since div $v = n$ everywhere in the zero set of v.

However, spacelike conformal vector fields never arise as the Dirac current of twistor spinors, see [Lei01]. *Causal* conformal vector fields with zeroes are studied in [Fra07]. In particular, one finds examples of Lorentzian manifolds admitting timelike or isotropic conformal vector fields with zero which are not locally conformally flat around the zero. These examples are much harder to construct than the above spacelike one. However, we can not clarify whether these vector fields also arise as Dirac currents of twistor spinors.

Finally, the shapes of the zero sets of general conformal vector fields in any metric signature are studied in [Der12, Der11]. One finds formulas that are conceptually very similar to the spinorial analogue (4.1).

5 Twistor Spinors in Low Dimensions

The aim of this chapter is the classification of pseudo-Riemannian geometries admitting twistor spinors in some low dimensions by making use of the theory developed so far. In fact, the results from the previous two chapters open a conceptual way to classify twistor spinors in arbitrary signature (p,q) and dimension $n = p + q$ once the orbit structure of $\Delta_{p,q}$ and $\Delta_{p+1,q+1}$ under the action of the respective spin groups is known. In this chapter we show how all information about possible local geometries admitting twistor spinors is encoded in these algebraic data. However, the mentioned orbit structure is only known for low values of p and q and we then apply this procedure to classify real twistor spinors in split signatures $(m, m-1)$ and (m, m) for $m \leq 7$.
Our new results obtained here add to other classification results for twistor spinors in low dimensions which are already known:

- [Bry00] classifies metrics admitting parallel spinor fields in small dimensions. It is moreover known that a Riemannian 3-manifold admitting a twistor spinor is conformally flat, and a Riemannian 4-manifold with twistor spinor is selfdual ([BFGK91]).
- In Lorentzian geometry, there is a classification of all local geometries admitting twistor spinors without zeroes and constant causal type of the associated conformal vector field V_φ for dimensions $n \leq 7$, which can be found in [Lei01, BL04].
- In signature $(2,2)$, anti-selfdual four manifolds with parallel real spinor have been studied in [Dun02]. Furthermore, [HS11a] presents a Fefferman construction which starts with a 2-dimensional projective structure and produces geometries carrying two pure spin tractors with nontrivial pairing which leads to $Hol(M,c) \subset SL(3,\mathbb{R}) \subset SO^+(3,3)$.
- [HS11b] investigates (real) generic twistor spinors in signature $(3,2)$ and $(3,3)$, being twistor spinors satisfying additionally that the constant (!) $\langle \varphi, D\varphi \rangle$ is nonzero (signature $(3,3)$ is also discussed in [Bry09]). They are shown to be in tight relationship to so called generic 2-distributions on 5-manifolds resp. generic 3-distributions on 6-manifolds, that means every generic twistor spinor φ gives rise to a generic distribution $\ker \varphi \subset TM$, and conversely, given a manifold with generic distribution, one can canonically construct a conformal structure admitting a twistor spinor, and these two constructions are inverse to each other. We will elaborate more on this case in the last section of this chapter.

In particular, we will study in this chapter the remaining non-generic cases of twistor spinors in signatures $(3,2)$ and $(3,3)$.

5.1 The relation between the orbit structure of the spinor module and local geometries

The purpose of this chapter is to classify geometries admitting *real* twistor spinors in low-dimensional split-signatures, i.e. $(p,q) = (m+1,m)$ or (m,m). In the latter case we may restrict our attention to twistor half-spinors since we are only interested in local considerations. Suppose we are given a complete list of orbit representatives $v_i \in \Delta^{\mathbb{R},(\pm)}_{p+1,q+1}$ for some index set I, i.e.

$$\cup_{i\in I} Spin^+(p+1,q+1)\cdot v_i = \Delta^{\mathbb{R},(\pm)}_{p+1,q+1},$$
$$Spin^+(p+1,q+1)\cdot v_i \cap Spin^+(p+1,q+1)\cdot v_j = \varnothing \text{ for } i \neq j,$$

with known stabilizer subgroups

$$Stab_{v_i} Spin^+(p+1,q+1) = \{g \in Spin^+(p+1,q+1) \mid g\cdot v_i = v_i\}.$$

Note that as $-1 \notin Stab_{v_i} Spin^+(p+1,q+1)$, the double covering $\lambda : Spin^+(p+1,q+1) \to SO^+(p+1,q+1)$ restricted to this stabilizer restricts to an isomorphism

$$\lambda : Stab_{v_i} Spin^+(p+1,q+1) \to \lambda(Stab_{v_i} Spin^+(p+1,q+1)).$$

Clearly, to each orbit we can associate the integer dim ker $v_i \in \{0,...,m\}$ which does not depend on the chosen orbit representative.

With these algebraic ingredients in mind, we now turn to geometry: Let $(M^{p,q},c)$ be a conformal spin manifold with nontrivial parallel spin tractor $\psi \in \Gamma(M,\mathcal{S})$. Via the holonomy principle, there is a unique orbit representative v_i corresponding to ψ [1]. As a direct consequence, we have that

- $k_\psi :=$ dim ker $\psi =$ dim ker v_i,
- $Hol(M,c) \subset \lambda(Stab_{v_i} Spin^+(p+1,q+1)) \subset SO^+(p+1,q+1)$ (up to conjugation).

Thus, in order to describe conformal structures of signature (p,q) admitting twistor spinors, one has to go through the list of orbit representatives v_i in $\Delta_{p+1,q+1}$ as these orbits precisely correspond to possible types of twistor spinors. If dim ker $v_i > 0$, the twistor spinor associated to ψ can locally be rescaled to a parallel spinor (Theorem 3.43). In any case, the stabilizer of v_i yields a reduction of conformal holonomy. We can then use Theorem 3.43 in order to derive further geometric consequences.
Moreover, the quantity $k_\psi =$ dim ker v_i also yields information about the zero set: Let $\varphi = \widetilde{\Phi}^g(proj^g_+\psi) \in$ ker P^g. Then by the previous chapter, $Z_\varphi \neq \varnothing$ is only possible if $k_\psi > 0$, and if $Z_\varphi \neq \varnothing$ it holds for its dimension as manifold by the proof of Proposition 4.18 that

$$\text{dim } Z_\varphi = k_\psi - 1 = \text{dim ker } v_i - 1. \tag{5.1}$$

[1] Equivalently, one can also consider ψ as $Spin^+(p+1,q+1)$-equivariant map $\widehat{\psi} : \overline{\mathcal{Q}}^1_+ \to \Delta_{p+1,q+1}$. As ψ is parallel, the image of this map is an orbit under the $Spin^+(p+1,q+1)$-action.

5.2 Orbit structure in low dimensional split signatures

Whenever possible, we list orbit representatives and their stabilizers without making use of an explicit realisation of an irreducible $Cl(p,q)$–representation, The following results are mainly due to [Igu70]. Equivalent results can also be found in [Bry00] up to dimension 8. However, reference [Igu70] uses a different realisation of the Clifford algebra $Cl(p,q)$ and its irreducible representations in split signatures. From the classification results in these references, it is then straightforward to extract a complete list of orbit representatives in $\Delta_{p,q}$ and associated stabilizer subgroups. For more details on these stabilizer groups we refer to the given references.

In signatures $(p,q) = (2,2),(3,2)$ and $(3,3)$, every nonzero (half-)spinor in $\Delta^{\mathbb{R}}_{p,q}$ is pure. Whence, its kernel under Clifford multiplication is maximal and the stabilizer is given by $R^+(p,q)$ (cf. 1.14).

The orbits of $\Delta^{\mathbb{R}}_{4,3}\backslash\{0\}$ under the action of $Spin^+(4,3)$ are precisely the level sets of $\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{R}}_{4,3}}$. A pure spinor is thus characterized by $\langle v,v\rangle_{\Delta^{\mathbb{R}}_{4,3}} = 0$ and $v \neq 0$ and its stabilizer is given by $R^+(4,3)$. By Proposition 1.16 the kernel of a non-pure spinor under Clifford multiplication is trivial. The stabilizers of these spinors are all conjugate and given by $G_{2,2}$.

In signature $(4,4)$ one has essentially the same orbit structure on the half spinor module, i.e. the orbits are given by the level sets of $\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{R}}_{4,4}}$ and the pure spinors constitute the null cone. Moreover, the stabilizer of a spinor with nonzero norm is given by $Spin^+(4,3) \subset Spin(4,4)$ [2].

Each level set of $\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{R}}_{5,4}}$ constitutes a single orbit under the $Spin^+(5,4)$-action. The stabilizer is given by $Spin^+(4,3) \subset Spin^+(4,4) \subset Spin^+(5,4)$. In particular, the image of this stabilizer under the double covering λ preserves a timelike line. In contrast to this, the null cone $\{v \in \Delta^{\mathbb{R}}_{5,4}\backslash\{0\} \mid \langle v,v\rangle_{\Delta^{\mathbb{R}}_{5,4}} = 0\}$ decomposes into two orbits: besides the orbit of pure spinors with stabilizer isomorphic to $R^+(5,4)$, there is an orbit of spinors with 1-dimensional kernel under Clifford multiplication.

$\Delta^{\mathbb{R},\pm}_{5,5}\backslash\{0\}$ decomposes into precisely two orbits under the $Spin^+(5,5)$-action. Besides the orbit of pure spinors with stabilizer $R^+(5,5)$ there is an orbit where every spinor has 1-dimensional kernel. In particular, the λ-image of a spinor lying in this orbit fixes an isotropic line.

In $\Delta^{\mathbb{R}}_{6,5}\backslash\{0\}$ one finds the orbit of pure spinors, one orbit whose elements have 2-dimensional kernel, one orbit whose elements have 1-dimensional kernel under Clifford multiplication and orbits whose elements have trivial kernel under Clifford multiplication. Among the last ones, there are two different types: There is one type where the λ-image of the stabilizer is isomorphic to $Sp(4) \ltimes N$ with N being some 14-dimensional group (for details cf. [Igu70]). This stabilizer acts reducible on $\mathbb{R}^{6,5}$ and preserves a 5-dimensional totally isotropic subspace. The stabilizers of other orbit representatives with trivial kernel are all conjugate to $SU(3,2) \subset SO(6,4) \subset SO(6,5)$.

[2] For the correct embedding $Spin(4,3) \hookrightarrow Spin(4,4)$ which has to be used in this situation, cf. [Kat99]

$\Delta^{\mathbb{R},\pm}_{6,6}\backslash\{0\}$ has the following orbit structure: There is the orbit of pure spinors with stabilizer $R^+(6,6)$ and an orbit whose elements have 2-dimensional kernel under Clifford multiplication. All other spinors have trivial kernel under Clifford multiplication. They split into two different types according to the action of the λ-image of their stabilizers: There is one orbit whose stabilizer subgroup acts reducible on $\mathbb{R}^{6,6}$ and preserves a maximally totally isotropic subspace of dimension 6. All other orbits with trivial kernel can be parametrized by one real parameter and have stabilizer subgroup isomorphic to $SU(3,3) \subset SO^+(6,6)$.

$\Delta^{\mathbb{R}}_{7,6}\backslash\{0\}$ decomposes into 20(!) different orbits under the action of $Spin^+(7,6)$. We do not go into detail and later wont analyse twistor spinors in signature $(6,5)$. In $\Delta^{\mathbb{R},\pm}_{7,7}\backslash\{0\}$ there are 9 different orbits. A complete description of both signatures can be found in [GE78].

5.3 Local geometric classification

With these preparations, it is now straightforward to give a complete list of local split-signature geometries admitting twistor spinors in low dimensions. We use the notation from section 5.1.

Theorem 5.1 *Let $(M^{p,q},c)$ be a conformal structure of split signature $(p,q)=(m,m)$ or $(m,m-1)$ with $m \leq 6$ and $(p,q) \neq (6,5)$. Assume that there exists a nontrivial real twistor (half)-spinor φ on M. Then exactly one of the following cases regarding the local geometry of (M,c) and the zero set structure of φ occurs:*

(p,q)	k_ψ	$Hol(M,c) \subset$	*local geometry in conformal class, local conformal behaviour of φ*	*type (Thm. 3.43)*	*dim* Z_φ (if $\neq \varnothing$)
(2,2)	3	$R^+(3,3)$	parallel pure spinor on Ricci-isotropic pseudo-2-Walker manifold with parallel, totally lightlike 2-form, local metric given by (3.38) ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	2
(3,2)	3	$R^+(4,3)$	parallel pure spinor on Ricci-isotropic pseudo-2-Walker manifold with parallel, totally lightlike 2-form, local metric given by (3.38) ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	2
	0	$G_{2,2}$	generic twistor spinor, i.e. $\langle \varphi, D^g\varphi \rangle \neq 0$, Geometry equivalently described in terms of generic 2-distribution ker $\varphi \subset TM$, cf. [HS11a]	2. or 3.	$\varnothing$

(3,3)	4	$R^+(4,4)$	parallel pure spinor on Ricci-isotropic pseudo-3-Walker manifold with parallel, totally lightlike 3-form, local metric given by (3.38) ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	3
	0	$Spin^+(4,3)$	generic twistor spinor, i.e. $\langle \varphi, D^g\varphi \rangle \neq 0$, Geometry equivalently described in terms of generic 3-distribution ker $\varphi \subset TM$, cf. [HS11a]	2. or 3.	$\varnothing$
(4,3)	4	$R^+(4,4)$	parallel pure spinor on Ricci-isotropic pseudo-3-Walker manifold with parallel, totally lightlike 3-form, local metric given by (3.38) ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	2
	1	fixes isotr. line	There is locally around each point off a singular set a neighbourhood U and metric $g \in c_{\vert U}$ such that (U,g) is Ricci-flat, φ is parallel on U wrt. g and $Hol(U,g) \subset G_{2,2}$. Such geometries are further discussed in [Kat99]	1.	0
	0	$Spin^+(4,3) \subset SO^+(4,4) \subset SO^+(5,4)$, fixes timelike vector	There is locally around each point off a singular set a neighbourhood U and an Einstein metric $g \in c_{\vert U}$ of positive scalar curvature. (U,g) has cone holonomy in $Spin^+(4,3)$ and on U, φ is a real Killing spinor.	2.b	$\varnothing$
(4,4)	5	$R^+(5,5)$	parallel pure spinor on Ricci-isotropic pseudo-4-Walker manifold with parallel, totally lightlike 4-form, local metric given by (3.38) ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	2
	1	fixes isotr. line	There is locally around each point off a singular set a neighbourhood U and metric $g \in c_{\vert U}$ such that (U,g) is Ricci-flat, φ is parallel on U wrt. g and $Hol(U,g) \subset Spin^+(4,3)$	1.	0
(5,4)	5	$R^+(6,5)$	parallel pure spinor on Ricci-isotropic pseudo-4-Walker manifold with parallel, totally lightlike 4-form, local metric given by (3.38) ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	2

	2	fixes lightlike plane	locally parallel spinor on conformally pure radiation metric with parallel rays as described in [LN12a]	1.	1
	1	fixes isotr. line	There is locally around each point off a singular set a neighbourhood U and metric $g \in c_{\mid U}$ such that (U,g) is Ricci-flat, φ is parallel on U wrt. g and $Hol(U,g) \subset Spin^+(4,3) \subset SO(5,4)$, i.e. one has a splitting $(U,g) \cong (U',g') \times (\mathbb{R}, -dt^2)$ and φ induces a parallel spinor on (U',g') with holonomy in $Spin^+(4,3) \subset SO^+(4,4)$.	1.	0
	0	$Sp(4) \ltimes N$, cf. [Igu70]	there exists locally a Ricci-isotropic pseudo-3-Walker metric in the conformal class on which φ is a nonparallel twistor spinor	2.a	$\varnothing$
	0	$SU(3,2) \subset SO^+(6,4)$	There is locally an Einstein-Sasaki metric (U,g) of negative scalar curvature in the conformal class on which φ decomposes into the sum of 2 imaginary Killing spinors	2.	$\varnothing$
(5,5)	6	$R^+(6,6)$	parallel pure spinor on Ricci-isotropic pseudo-5-Walker manifold with parallel, totally lightlike 5-form, local metric given by (3.38) ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	2
	2	Stab. lightlike plane	locally parallel spinor on conformally pure radiation metric with parallel rays as described in [LN12a]	1.	1
	0	cf. [Igu70], fixes 6-dim. lightlike space	there exists locally a Ricci-isotropic pseudo-5-Walker metric in the conformal class on which φ is a nonparallel twistor spinor	2.	0
	0	$SU(3,3)$	There is locally a Fefferman metric in the conformal class on which φ is a nonparallel twistor spinor	2. or 3.	$\varnothing$
(6,6)	7	$R^+(7,7)$	parallel pure spinor on Ricci-isotropic pseudo-6-Walker manifold with parallel, totally lightlike 6-form, local metric given by (3.38) ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	2
	3	fixes lightlike subspace of dim. 3	parallel pure spinor on Ricci-isotropic pseudo-2-Walker manifold ker φ is *globally* integrable distribution on $\widetilde{M} \subset M$ open, dense	1.	2.

	1	fixes isotr. line	There exists locally around each point off a singular set a neighbourhood $U \subset M$ and a local metric $g \in c_U$ such that φ is a parallel spinor on the Ricci-flat space (U, g). There are two different possibilities: Either fixes a maximally isotropic subspace, or $Hol(U, g) \subset SU(3,3)$, i.e. (U, g) is pseudo-Kähler.	1.	0
	0	5 different possible stabilizer types which can be found in [GE78]	*general solution unknown*	?	$\varnothing$

Proof. The proof is a direct application of the procedure described at the beginning of this chapter. In the notation section 5.1, every twistor spinor $\varphi \in \Gamma(M^{p,q}, S^g_{\mathbb{R}})$ is associated with a unique $Spin^+(p+1, q+1)$-orbit type, which have been classified in the previous section. Let v_i be a representative of this orbit type. Then the possible values for dim ker ψ = dim ker v_i follow directly from this orbit type classification. Moreover, the stabilizer of v_i yields a reduction of conformal holonomy. These conformal holonomy reductions are then directly linked to local geometric structures in the conformal class and local behaviour of the spinor φ by our main Theorem 3.43 and the results from section 3.3. Moreover, Proposition 3.34 describes the type of parallel spinor one has in case dim ker $\psi > 0$. For the last case in signature $(5,4)$ we moreover used that the conformal holonomy reduces to that of the metric cone for the Einstein metric in the conformal class and that unitary holonomy of the cone is equivalent to a Sasakian structure on the base. This yields the third column of the table. The possible dimensions of the zero set follow from (5.1). □

Remark 5.2 We observe from Theorem 5.1 that in all studied split signatures there is a one-to-one correspondence between possible shapes of the zero set of a twistor (half-) spinor and local geometries off the zero set. This clearly generalizes Theorem 4.2 and Proposition 4.21 to cases where the index is greater than 2. In the cited statements we have observed that in case of an *isolated* zero and $p = 1, 2$ one always has a local splitting off the zero set. However, Theorem 5.1 tells us that this does not hold for arbitrary signatures. For example, in signature $(4,4)$ the local geometry of a manifold $(U, [g])$ admitting a twistor spinor with isolated zero is of exceptional metric holonomy $Hol(U, g) \subset Spin^+(4,3) \subset SO^+(4,4)$ and does not need to act reducible.

Remark 5.3 Together with the results from [BL04] for signature $(2,1)$, we obtain that for all split signatures (m, m) or $(m+1, m)$ with dimension of the manifold ≤ 8 the twistor equation always reduces to the equation for parallel spinors or Killing spinors after a local conformal change, except the generic cases in signatures $(3,2)$ and $(3,3)$.

Remark 5.4 We do not know whether there are nontrivial examples for all types of twistor spinors appearing in Theorem 5.1.

5.4 Real twistor spinors in signatures (3,2) and (3,3)

Let us elaborate on the results from the previous section for twistor spinors in signatures $(3,2)$ and $(3,3)$ in more detail.

Let $(M,[g])$ be a conformal spin manifold of signature $(3,2)$. We work with the real spinor bundle $S^g = S^g_{\mathbb{R}}(M)$ and real spin tractor bundle $\mathcal{S} = \mathcal{S}_{\mathbb{R}}(M)$.
Let $\psi \in \Gamma(M,\mathcal{S})$ be a nontrivial parallel spin tractor. The quantity $\langle \psi,\psi\rangle_{\mathcal{S}} \in \mathbb{R}$ is constant over M. The orbit structure discussion from section 5.2 reveals that

$$\langle \psi,\psi\rangle_{\mathcal{S}} = \begin{cases} \text{const.} \neq 0 & \text{iff dim ker } \psi = 0, \\ 0 & \text{iff dim ker } \psi = 3. \end{cases}$$

Let $\varphi = \widetilde{\Phi}^g(proj^g_+\psi) \in \ker P^g$ be the twistor spinor associated to ψ. A direct application of the scalar product formula (1.12) shows that (cf. also [HS11a])

$$\langle \psi,\psi\rangle_{\mathcal{S}} = d \cdot \langle \varphi, D^g\varphi\rangle_{S^g},$$

where d is a nonzero constant. In particular, the right side is constant and does not depend on $g \in [g]$. Consequently, real twistor spinors in signature (3,2) fall in two disjoint classes:

In the first case, it is $\langle \varphi, D^g\varphi\rangle_{S^g} \neq 0$. It follows that dim ker $\psi = 0$ and $Hol(M,c) \subset G_{2,2}$. Such twistor spinors are called generic and studied in [HS11b]: For these spinors, the distribution $H :=$ ker $\varphi \subset TM$ is of constant rank 2 and turns out to be a generic 2-distribution, i.e. $[H,[H,H]] = TM$. On the other hand, using the general machinery of parabolic geometries from [CS09], [HS11b] shows that given any 5-dimensional manifold M admitting an oriented, generic 2-distribution H, there is a canonical (Fefferman-type) construction of a conformal structure[3] $[g]$ of signature $(3,2)$ on M admitting a twistor spinor $\varphi \in \Gamma(M,S^g)$ with $H =$ ker φ.

In contrast, the case of non-generic real twistor spinors in signature $(3,2)$, i.e. those satisfying $\langle \varphi, D^g\varphi\rangle_{S^g} = 0$ is covered by the preceding discussion from this chapter: In this case, we have that dim ker $\psi = 3$. Consequently, Proposition 3.28 applies, yielding integrability of the distribution ker φ off a singular set. Moreover, by the same Proposition the spinor φ is locally equivalent to a parallel spinor. As every nonzero real spinor in signature $(3,2)$ is pure, Theorem 3.36 applies and yields a local normal form for the metric. We summarize:

Theorem 5.5 *Let $(M,[g])$ be a conformal spin manifold of signature $(3,2)$ admitting a real twistor spinor $\varphi \in \Gamma(S^g)$. Then the function $\langle \varphi, D^g\varphi\rangle_{S^g}$ is constant and the value of this constant does not depend on the chosen metric in $[g]$. We distinguish the following cases:*

1. *$\langle \varphi, D^g\varphi\rangle_{S^g} \neq 0$. In this case, $Hol(M,[g]) \subset G_{2,2}$, the 2-dimensional distribution ker $\varphi \subset TM$ is generic and the whole conformal structure can be recovered from it.*

[3] A more explicit way of defining the conformal structure in terms of a generic 2-distribution in dimension 5 is presented in [Nur05].

2. *$\langle\varphi, D^g\varphi\rangle_{S^g} = 0$. In this case, $Hol(M,[g])$ fixes a 3-dimensional totally lightlike subspace, there is an open, dense subset $\widetilde{M} \subset M$ on which the distribution ker φ is of constant rank 2 and integrable. Moreover, φ is locally conformally equivalent to a parallel pure spinor wrt. a local metric from Theorem 3.36 which lies in the conformal class.*

A completely analogous procedure can be carried out for real twistor half-spinors φ in signature $(3,3)$. Again, the value $\langle\varphi, D^g\varphi\rangle_{S^g}$ is constant over M and given by the length of the associated spin tractor ψ. The case that this is nonzero has been studied in [HS11b]. $\langle\varphi, D^g\varphi\rangle_{S^g}$ being zero is as in the $(3,2)$-case equivalent to say that the associated spin tractor ψ has zero length, i.e. it is pure as follows from the orbit discussion in section 5.2. Consequently dim ker $\psi = 4$. Thus, Proposition 3.28 applies to ψ. We get that φ is locally conformally equivalent to a real, parallel half-spinor in signature $(3,3)$ which is pointwise pure by the orbit discussion from section 5.2. This gives the second part of the next statement. The first part in the following Theorem is discussed in [HS11b]:

Theorem 5.6 *Let $(M,[g])$ be a conformal spin manifold of signature $(3,3)$ admitting a real twistor half-spinor $\varphi \in \Gamma(S^g_{\pm})$. Then the function $\langle\varphi, D^g\varphi\rangle_{S^g}$ is constant and the value of this constant does not depend on the chosen metric in $[g]$. We distinguish the following cases:*

1. *$\langle\varphi, D^g\varphi\rangle_{S^g} \neq 0$. In this case, $Hol(M,[g]) \subset Spin^+(4,3) \subset Spin^+(4,4)$, the 3-dimensional distribution H =ker $\varphi \subset TM$ is generic, i.e. $[H,H] = TM$ and the whole conformal structure can be recovered from it by a Fefferman construction.*
2. *$\langle\varphi, D^g\varphi\rangle_{S^g} = 0$. In this case, $Hol(M,[g])$ fixes a 4-dimensional totally lightlike subspace, there is an open, dense subset $\widetilde{M} \subset M$ on which the distribution ker φ is of constant rank 3 and integrable. Moreover, φ is locally conformally equivalent to a parallel pure spinor wrt. a local metric from Theorem 3.36 which lies in the conformal class.*

6 Tractor Conformal Superalgebras in Lorentzian Signature

In the previous chapters we studied the question: If there *exists* a twistor spinor on a conformal manifold (M, c), what can be said about the local conformal geometry of M ? This problem admits a natural generalization: In physics, one is often not only interested in the existence of solutions of certain spinor field equations, but wants to relate the existence of a certain number of maximally linearly independent solutions to local geometric structures. There are some important examples: [FMS05] shows that supersymmetric M-theory backgrounds admitting *enough* supersymmetries, which are defined as Killing spinors with respect to a suitable connection, are locally homogeneous. [AC08] studies the relation between the existence of a certain number of parallel-, Killing- and twistor spinors and underlying local geometries. We are going to recall some of these results in more detail in this chapter. Consequently, the generalization of the above question goes as follows:

Given a conformal manifold (M, c) where we know the *algebraic structure* of all conformal symmetries, what can then be said about the local geometry of M ?

Let us first make precise what we mean by the algebraic structure of all conformal symmetries. The key object is the following:

Definition 6.1 *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$−graded $\mathbb{K}$−vector space. For a homogeneous element $X \in \mathfrak{g}$, we let $|X| := i$ if $X \in \mathfrak{g}_i$. $\mathfrak{g}$ together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called a $(\mathbb{K}-)$superalgebra if*

1. $[\cdot,\cdot] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$,
2. *For homogeneous elements $X, Y \in \mathfrak{g}$ it holds that* $[X, Y] = -(-1)^{|X||Y|}[Y, X]$.

If moreover the Jacobi identity

$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]] \tag{6.1}$$

holds for all homogeneous elements, we call $\mathfrak{g}$ a Lie superalgebra.

Superalgebras and their classifications naturally appear in the context of supersymmetry in physics (cf. [Nah78]), and their classification theory is also of interest for purely mathematical reasons. Definition 6.1 naturally applies to pseudo-Riemannian conformal structures $(M^{p,q}, [g])$ as follows: On the space $\mathfrak{X}^{nc}(M) \oplus \ker P^g$ of *normal* conformal vector fields and twistor spinors we introduce brackets by setting:

$$\begin{aligned} [V,W] &:= [V,W]_{\mathfrak{X}(M)}, \\ [V,\varphi] &:= V\circ\varphi, \\ [\varphi,V] &:= -V\circ\varphi, \\ [\varphi_1,\varphi_2] &:= V_{\varphi_1,\varphi_2}, \end{aligned} \tag{6.2}$$

where $V,W \in \mathfrak{X}^{nc}(M), \varphi \in \ker P^g$. $V\circ\varphi$ is the spinorial Lie derivative which was introduced in [Kos72], for instance, and which we review in this chapter. It is proved in [Raj06] that $\mathfrak{g} := \mathfrak{X}^{nc}(M) \oplus \ker P^g$ together with these brackets is a superalgebra which is in general *no* Lie superalgebra. It has earlier been observed in [Hab96] that also the space $\mathfrak{g}^{ec} := \mathfrak{X}^c(M) \oplus \ker P^g$ of conformal vector fields and twistor spinors equipped with the same brackets turns out to be a superalgebra which in general is no Lie superalgebra. We will discuss later why we choose only normal conformal vector fields in the even part. From now on let us assume that $(M,[g])$ is a Lorentzian conformal structure, i.e. $p = 1$. This ensures that the bracket (6.2) makes the superalgebra structure become nontrivial.

The mentioned constructions of conformal superalgebras involving twistor spinors all fix a metric in the conformal class. In contrast to this, our aim is the construction of a superalgebra canonically associated to a conformal spin structure by making use of conformal tractor calculus. As we shall see, this approach reproduces the above constructions from [Raj06, Hab96] when we fix a metric in the conformal class, and thus it yields an equivalent description of the conformal symmetry superalgebra. However, the tractor approach as presented here has the advantage of giving conditions in terms of conformal holonomy exhibiting when the construction actually leads to a Lie superalgebra. This turns out to be very useful in respect of the question raised above.

In this chapter, we thus define a superalgebra $\mathfrak{g}$ canonically associated to a Lorentzian conformal spin manifold (M,c) using tractor calculus, then describe $\mathfrak{g}$ wrt. some fixed $g \in c$, study the relation between properties of $\mathfrak{g}$ and special geometries in the conformal class c, and sketch a generalization to non-Lorentzian signatures.

6.1 The general construction of tractor conformal superalgebras

Let $(M^{1,n-1},c)$ be a Lorentzian conformal spin manifold. In this chapter, when dealing with spinor- and spin tractor bundles, we always mean the *complex ones*, i.e. $\mathcal{S}(M) = \mathcal{S}_{\mathbb{C}}(M)$ or $S^g(M) = S^g_{\mathbb{C}}(M)$, obtained as associated vector bundles to $\mathcal{Q}^1_+$ or $\mathcal{Q}^g_+$ using $\Delta^{\mathbb{C}}_{2,n}$ or $\Delta^{\mathbb{C}}_{1,n-1}$, respectively. We present a canonical construction

$(M^{1,n-1},c)$		$\to \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$(real) Superalgebra,
$(M^{1,n-1},c)$	with *special* holonomy	$\to \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *Lie* Superalgebra.

Clearly, the even part should correspond to infinitesimal conformal symmetries, i.e. conformal vector fields. As elaborated in detail in [Raj06] it is natural to consider only *normal* conformal vector fields, i.e. duals of normal conformal Killing 1-forms. A superalgebra construction which does not need the vector fields to be normal is presented in [Hab96].[1]

[1] We will comment further on the difference between conformal and normal conformal vector fields in Remark 6.13 and Section 7.7.

From Remark 3.9 it is clear that for fixed $g \in c$ normal conformal vector fields are in one-to one correspondence to parallel tractor 2-forms. We thus set

$$\mathfrak{g}_0 := Par\left(\Lambda^2_{\mathcal{T}}(M), \nabla^{nc}\right) \subset \Omega^2_{\mathcal{T}}(M).$$

Furthermore, as motivated from physics literature, see [MH13] and the previous attempts to construct a conformal superalgebra, the elements of the odd part should correspond to the spinorial conformal analogue of infinitesimal conformal transformations being nothing but twistor spinors [2]. In light of the equivalent characterization of twistor spinors in terms of parallel spin tractors from Theorem 3.3, it is reasonable to set

$$\mathfrak{g}_1 := Par\left(\mathcal{S}(M), \nabla^{nc}\right) \subset \Gamma\left(\mathcal{S}(M), \nabla^{nc}\right).$$

We now introduce natural brackets which make $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ become a superalgebra:

For the even-even bracket, note that pointwise application of (1.2) yields a canonical isomorphism

$$\begin{aligned} \tau : \Omega^2_{\mathcal{T}}(M) &\to \mathfrak{so}(\mathcal{T}(M), \langle\cdot,\cdot\rangle_{\mathcal{T}}), \\ \alpha &\mapsto \alpha_E,\ \alpha_E(X) := (X \lrcorner \alpha)^\sharp. \end{aligned} \tag{6.3}$$

$\mathfrak{so}(\mathcal{T}(M), \langle\cdot,\cdot\rangle_{\mathcal{T}})$ carries the pointwise defined usual Lie bracket of endomorphisms. We use τ, to carry this structure over to α, i.e. we set for $\alpha, \beta \in \mathfrak{g}_0$

$$[\alpha, \beta] := \tau^{-1}\left(\alpha_E \circ \beta_E - \beta_E \circ \alpha_E\right).$$

Moreover, ∇^{nc} induces a covariant derivative ∇^{nc} on $\mathfrak{so}(\mathcal{T}(M), \langle\cdot,\cdot\rangle_{\mathcal{T}})$ in a natural way.

Proposition 6.2 *For $\alpha, \beta \in \mathfrak{g}_0$ we have that also $[\alpha, \beta] \in \mathfrak{g}_0$.*

Proof. We first show that that $\alpha \in Par\left(\Lambda^2_{\mathcal{T}}(M), \nabla^{nc}\right) \Leftrightarrow \alpha_E \in Par\left(\mathfrak{so}(\mathcal{T}(M), \langle\cdot,\cdot\rangle_{\mathcal{T}}), \nabla^{nc}\right)$: Let $X \in \mathfrak{X}(M)$, $x \in M$ and let $(v_0, ..., v_{n+1})$ be a local frame in $\mathcal{T}M$ which is parallel in x wrt. ∇^{nc}. We have for $i \in \{0, ..., n+1\}$ at x:

$$\left(\nabla^{nc}_X \alpha_E\right)(v_i) = \nabla^{nc}_X\left(\alpha_E(v_i)\right) = \nabla^{nc}_X (v_i \lrcorner \alpha)^\sharp = \left(\nabla^{nc}_X (v_i \lrcorner \alpha)\right)^\sharp = (v_i \lrcorner \nabla^{nc}_X \alpha)^\sharp,$$

which proves this claim. Thus, it suffices to check that for $\alpha, \beta \in \mathfrak{g}_0$ also $[\alpha_E, \beta_E]_{\mathfrak{so}} \in Par\left(\mathfrak{so}(\mathcal{T}(M), \langle\cdot,\cdot\rangle_{\mathcal{T}}), \nabla^{nc}\right)$. We compute with the same notations as above at x:

$$\begin{aligned} \left(\nabla^{nc}_X \left([\alpha_E, \beta_E]_{\mathfrak{so}}\right)\right)(v_i) &= \nabla^{nc}_X\left([\alpha_E, \beta_E]_{\mathfrak{so}}(v_i)\right) - [\alpha_E, \beta_E]_{\mathfrak{so}}\left(\nabla^{nc}_X v_i\right) \\ &= \nabla^{nc}_X\left(\alpha_E(\beta_E(v_i)) - \beta_E(\alpha_E(v_i))\right) = \alpha_E(\beta_E(\nabla^{nc}_X v_i)) - \beta_E(\alpha_E(\nabla^{nc}_X v_i)) \\ &= 0 \end{aligned}$$

This proves the Proposition. □

Clearly, $\mathfrak{g}_0$ now becomes a Lie algebra in the usual sense. We shall show in the next section that the chosen bracket is *the right one* in the sense that if α, β are considered as normal conformal vector fields for some fixed $g \in c$ by means of $\left(proj^g_{\Lambda,+}\alpha\right)^\sharp$, then $[\cdot,\cdot]$ translates into the usual Lie bracket of vector fields.

[2] This can also be made precise in the language of supermanifolds as done in [Kli05], for example.

As a next step we define the odd-odd bracket, which by definition has to be a symmetric bilinear map $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$. A nontrivial way to obtain a parallel tractor 2-form from two parallel spin tractors is given by the parallel tractor form from (2.5), i.e.

$$[\cdot,\cdot] : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0 \ , \ (\psi_1, \psi_2) \mapsto \alpha^2_{\psi_1,\psi_2}.$$

In signature $(2,n)$, the form $\alpha^2_{\psi_1,\psi_2}$ is given as follows: One observes that $\langle \alpha\cdot\psi, \psi\rangle_{\Delta^{\mathbb{C}}_{2,n}} \in i\mathbb{R}$ for $\psi \in \Delta^{\mathbb{C}}_{2,n}, \alpha \in \Lambda^2_{2,n}$. (3.2) thus yields that

$$\langle \alpha^2_{\psi_1,\psi_2}, \alpha\rangle_{\mathcal{T}} = \text{Im } \langle \alpha \cdot \psi_1, \psi_2\rangle_{\mathcal{S}}, \ \alpha \in \Omega^2_{\mathcal{T}}(M). \tag{6.4}$$

$\alpha^2_{\psi_1,\psi_2}$ is then symmetric in ψ_1 and ψ_2.
It remains to introduce an even-odd-bracket. We set

$$[\cdot,\cdot] : \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1 \ , \ (\alpha, \psi) \mapsto \frac{1}{2}\alpha \cdot \psi.$$

The meaning of the factor $\frac{1}{2}$ will become clear in a moment. It follows directly from (2.5) that this map is well-defined, i.e. the image lies again in $\mathfrak{g}_1$. Moreover, in order to obtain the right symmetry relations, we must set

$$[\cdot,\cdot] : \mathfrak{g}_1 \times \mathfrak{g}_0 \to \mathfrak{g}_1 \ , \ (\psi, \alpha) \mapsto -\frac{1}{2}\alpha \cdot \psi.$$

The discussion shows that with these choices of $\mathfrak{g}_0, \mathfrak{g}_1$ and definitions of the brackets, we have associated a nontrivial (real) conformal superalgebra to the conformal structure (where $\mathfrak{g}_1$ is considered as a *real* vector space).

Definition 6.3 *The (real) superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ associated to $(M^{1,n-1}, c)$ is called the* tractor conformal superalgebra *(associated to (M,c)).*

It is natural to ask under which circumstances the construction produces a *Lie* superalgebra, i.e. we have to check the four Jacobi identities from (6.1). As $\mathfrak{g}_0$ is a Lie algebra in its own right, the even-even-even Jacobi identity is always satisfied.

Proposition 6.4 *The tractor conformal superalgebra associated to a Lorentzian conformal spin manifold satisfies the even-even-odd and the even-odd-odd Jacobi identity.*

Proof. By (6.1) we have to check that

$$[\alpha, [\beta, \psi]] \overset{!}{=} [[\alpha, \beta], \psi] + [\beta, [\alpha, \psi]] \ \forall \alpha, \beta \in \mathfrak{g}_0, \psi \in \mathfrak{g}_1,$$

which by definition of the brackets is equivalent to showing that

$$2 \cdot [\alpha, \beta] \cdot \psi \overset{!}{=} \alpha \cdot \beta \cdot \psi - \beta \cdot \alpha \cdot \psi,$$

being a purely algebraic identity at each point. Whence, we may for the proof assume that $\alpha, \beta \in \Lambda^2_{2,n}$ and $\psi \in \Delta^{\mathbb{C}}_{2,n}$. With respect to the standard basis of $\mathbb{R}^{2,n}$ we express

$$\alpha = \sum_{i<j} \epsilon_i \epsilon_j \alpha_{ij} e_i^\flat \wedge e_j^\flat \Rightarrow \alpha_E = \sum_{i<j} \epsilon_i \epsilon_j \alpha_{ij} E_{ij} \text{ and } \beta = \sum_{k<l} \epsilon_k \epsilon_l \beta_{kl} e_k^\flat \wedge e_l^\flat \Rightarrow \beta_E = \sum_{k<l} \epsilon_k \epsilon_l \beta_{kl} E_{kl}.$$

This shows that

$$2\cdot[\alpha,\beta]\cdot\psi = \tau^{-1}\left([\alpha_E,\beta_E]_{\mathfrak{so}(2,n)}\right) = \sum_{i<j}\sum_{k<l}\epsilon_i\epsilon_j\epsilon_k\epsilon_l\alpha_{ij}\beta_{kl}\tau^{-1}\left(2\cdot[E_{ij},E_{kl}]_{\mathfrak{so}(2,n)}\right)\cdot\psi$$
$$= \sum_{i<j}\sum_{k<l}\epsilon_i\epsilon_j\epsilon_k\epsilon_l\alpha_{ij}\beta_{kl}[e_ie_j,e_ke_l]_{\mathfrak{spin}(2,n)}\cdot\psi = (\alpha\cdot\beta-\beta\cdot\alpha)\cdot\psi.$$

The even-odd-odd Jacobi identity is by polarization equivalent to $[\alpha,[\psi,\psi]] = [[\alpha,\psi],\psi]+[\psi,[\alpha,\psi]]$ for all $\alpha\in\mathfrak{g}_0$ and $\psi\in\mathfrak{g}_1$. By definition of the brackets, we have to show that

$$\left[\alpha_E,\left(\alpha^2_\psi\right)_E\right]_{\mathfrak{so}(\mathcal{T}(M))} \overset{!}{=} \left(\frac{1}{2}\alpha^2_{\alpha\cdot\psi,\psi}+\frac{1}{2}\alpha^2_{\psi,\alpha\cdot\psi}\right)_E = \left(\alpha^2_{\alpha\cdot\psi,\psi}\right)_E. \tag{6.5}$$

Again, this is pointwise a purely algebraic identity. Whence, it suffices to prove it for $\alpha\in\Lambda^2_{2,n}$ and $\psi\in\Delta^{\mathbb{C}}_{2,n}$. With respect to the standard basis of $\mathbb{R}^{2,n}$, we write α and α_E as above. Inserting the definition of α^2_ψ leads to

$$\left[\alpha_E,\left(\alpha^2_\psi\right)_E\right] = \sum_{i<j}\sum_{k<l}\epsilon_i\epsilon_j\epsilon_k\epsilon_l\alpha_{ij}\cdot\mathrm{Im}\left(\langle e_k\cdot e_l\cdot\psi,\psi\rangle_{\Delta^{\mathbb{C}}_{2,n}}\right)\cdot[E_{ij},E_{kl}], \tag{6.6}$$

whereas the right-hand side of (6.5) is by definition given by

$$\left(\alpha^2_{\alpha\cdot\psi,\psi}\right)_E = \sum_{k<l}\epsilon_k\epsilon_l\mathrm{Im}\left(\langle e_k\cdot e_l\cdot\alpha\cdot\psi,\psi\rangle_{\Delta^{\mathbb{C}}_{2,n}}\right)\cdot E_{kl}$$
$$= \sum_{i<j}\sum_{k<l}\epsilon_i\epsilon_j\epsilon_k\epsilon_l\alpha_{ij}\cdot\mathrm{Im}\left(\langle e_k\cdot e_l\cdot e_i\cdot e_j\cdot\psi,\psi\rangle_{\Delta^{\mathbb{C}}_{2,n}}\right)\cdot E_{kl}. \tag{6.7}$$

Using the algebra relations for $\mathfrak{so}(2,n)$ from (1.1), it is not difficult to show that every summand in (6.6) shows up also in (6.7) and vice versa:

- Consider summands with i,j,k,l pairwise distinct. Clearly, they vanish in (6.6). On the other hand, $\langle e_k\cdot e_l\cdot e_i\cdot e_j\cdot\psi,\psi\rangle_{\Delta^{\mathbb{C}}_{2,n}}\in\mathbb{R}$, i.e. the summands also vanish in (6.7).
- Consider summands with $i=k, j=l$. Again, they vanish in (6.6). In (6.7), these summands are proportional to $\langle\psi,\psi\rangle_{\Delta^{\mathbb{C}}_{2,n}}\in\mathbb{R}$, so the imaginary part vanishes.
- Consider summands in (6.7) with $i=k$ and $j\neq l$. They lead to the expression $-\epsilon_j\epsilon_l\alpha_{ij}\mathrm{Im}\left(\langle\epsilon_i\cdot e_j\cdot e_l\cdot\psi,\psi\rangle_{\Delta^{\mathbb{C}}_{2,n}}\right)E_{il}$. In (6.6), these summands can be found for choosing $j=k$ and $i\neq l$ for which we get $[E_{ij},E_{kl}] = -\epsilon_jE_{il}$, and thus the summand $-\epsilon_j\epsilon_l\alpha_{ij}\mathrm{Im}\left(\langle\epsilon_i\cdot e_j\cdot e_l\cdot\psi,\psi\rangle_{\Delta^{\mathbb{C}}_{2,n}}\right)E_{il}$ also shows up in (6.6). The remaining cases are equivalent to this one after permuting the indices.

Consequently, the two sums are identical and (6.5) holds. □

In contrast to that, the remaining Jacobi identity does not hold in general as we shall later see for concrete examples. Under certain restrictions on the conformal holonomy representation, we can however show that all Jacobi identities hold.

Theorem 6.5 *Suppose that the conformal holonomy representation of (M,c) satisfies the following: There exists for $x\in M$* ***no*** *(possibly trivial) $m-$dimensional Euclidean subspace $E\subset\mathcal{T}_x(M)\cong\mathbb{R}^{2,n}$ such that both*

1. *The action of $Hol_x(M,c)$ fixes E.*

2. *$E^\perp$ is even-dimensional and on $E^\perp$, $Hol_x(M,c)_{E^\perp} := \{A_{|E^\perp} \mid A \in Hol_x(M,c)\} \subset SO^+(E^\perp) \cong SO^+(2,n-m)$ is conjugate to a subgroup of $SU(1,\frac{n-m}{2}) \subset SO(2,n-m)$.*

Then the tractor conformal superalgebra $\mathfrak{g}$ satisfies the odd-odd-odd Jacobi identity, and thus carries the structure of a Lie superalgebra.

Remark 6.6 Before we go in the details of the proof, we remark that geometries admitting twistor spinors and which do **not** satisfy the conditions from Theorem 6.5 are well-understood: By Remark 1.25 and Theorem 3.32 they correspond to the cases $3.(a) - 3.(c)$ mentioned in Theorem 3.32, being Fefferman metrics, Lorentzian Einstein Sasaki manifolds or local splittings $g_1 \times g_2 \in [g]$ where g_1 is a Lorentzian Einstein Sasaki metric and g_2 is a Riemannian Einstein metric of positive scalar curvature. Thus, Theorem 6.5 can be rephrased in more geometric terms by saying that if none of these three special geometries lies in the conformal class of the metric, one obtains a conformal tractor Lie superalgebra.
This is in accordance with other observations in the literature (cf. [MH13]). Namely it is known that for the mentioned special geometries one has to include further symmetries in the algebra in order to obtain a conformal Lie superalgebra to which we will come back later.

Proof. As a first step, we show that under the assumptions, $\psi \in \mathfrak{g}_1 \Rightarrow \ker\psi \neq \{0\}$: For $\psi \in \mathfrak{g}_1$, the parallel tractor 2-form α^2_ψ must up to conjugation be one of the four generic types from the list of Remark 1.25. However, Remark 1.25 also shows that due to their stabilizers (which are maximal) under the $O(2,n)$-action, types 3. and 4. contradict our assumptions. Whence α^2_ψ is of type 1. or 2. But by Lemma 1.24, this implies that $\dim \ker\psi \in \{1,2\}$.
We now verify the odd-odd-odd Jacobi identity: By a standard polarization argument, this is equivalent to show that $[\psi,[\psi,\psi]] = 0$ for all $\psi \in \mathfrak{g}_1$. By definition of the brackets, this precisely says that

$$\alpha^2_\psi \cdot \psi \overset{!}{=} 0.$$

However, as $\ker\psi \neq \{0\}$, Lemma 1.24 yields that $\alpha^2_\psi = l^\flat \wedge r^\flat$, where $l \in \ker\psi$ and r is orthogonal to l. It follows that $\alpha^2_\psi \cdot \psi = -r \cdot l \cdot \psi = 0$. This proves the remaining Jacobi identity and the Theorem. □

Remark 6.7 Note that our construction of a conformal tractor superalgebra is the conformal analogue of the following metric construction: Given a Lorentzian spin manifold (M,g), let $\mathfrak{h}_0$ denote the space of parallel vector fields, and $\mathfrak{h}_1$ denote the space of parallel, complex spinor fields on (M,g). We equip $\mathfrak{h}_0$ with the (trivial) Lie bracket of vector fields, and for $X \in \mathfrak{h}_0, \varphi_1, \varphi_2 \in \mathfrak{h}_1$ it is obviously well-defined to set

$$\begin{aligned}[X,\varphi_1] &:= X \cdot \varphi_1 \in \mathfrak{h}_1,\\ [\varphi_1,\varphi_2] &:= V_{\varphi_1,\varphi_2} \in \mathfrak{h}_0.\end{aligned}$$

These brackets have the right symmetry properties which turn $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ into a superalgebra. The odd-odd-odd identity is now equivalent to

$$V_\varphi \cdot \varphi \stackrel{!}{=} 0 \text{ for all } \varphi \in \mathfrak{h}_1. \tag{6.8}$$

By Lemma 1.24 this holds if and only if V_φ is lightlike for every $\varphi \in \mathfrak{h}_1$. By a well-known result from [Lei01], V_φ is always causal. Whence, if V_φ is not lightlike there is by the holonomy principle a (local) splitting $(M,g) \cong (\mathbb{R}, -dt^2) \times (N,h)$. Thus, if we assume that $Hol(M,g)$ does not fix any timelike vector, (6.8) holds. This is the analogue to Theorem 6.5. However, note that the above algebra does not satisfy the even-even-odd Jacobi identity.
The construction of Killing superalgebras for Riemannian or Lorentzian manifolds using the cone construction where the even part consists of Killing vector fields and the odd part of geometric Killing spinors is discussed in [Far99], for instance. In case of an Einstein metric in the conformal class this is equivalent to our tractor construction as in this case all conformal holonomy computations restrict to considerations on the metric cone, see (3.15).

Remark 6.8 One can complexify the even part $\mathfrak{g}_0$ in order to obtain a complex superalgebra. Furthermore, the construction of a *real* tractor conformal superalgebra can completely analogous be carried out with *real* spinors. One then has to make the obvious modifications, i.e. define $\alpha^2_{\psi_1,\psi_2}$ without the imaginary part from (6.4). Note that by section 1.3 we have $\langle \psi, \psi \rangle_{\Delta^{\mathbb{R}}_{2,n}} = 0 \ \forall \psi \in \Delta^{\mathbb{R}}_{2,n}$. One obtains the same results, i.e. all Jacobi identities except the odd-odd-odd one are always satisfied. However, as we are later dealing with tractor conformal superalgebras for twistor spinors on Fefferman spaces, cf. [Bau99], it seems more appropriate to work with complex quantities in this chapter.

6.2 Description of the tractor conformal superalgebra with respect to a metric

We have already observed in Remark 3.9 and Theorem 3.3 that fixing a metric $g \in c$ leads to canonical isomorphisms

$$\begin{aligned} i_0 : \mathfrak{g}_0 &= Par\left(\Lambda^2_{\mathcal{T}}(M), \nabla^{nc}\right) \to \mathfrak{X}^{nc}(M), & \alpha &\mapsto V_\alpha := \left(proj^g_{\Lambda,+}(\alpha)\right)^\sharp, \\ i_1 : \mathfrak{g}_1 &= Par\left(\mathcal{S}(M), \nabla^{nc}\right) \to \ker P^g, & \psi &\mapsto \varphi := \widetilde{\Phi}^g(proj^g_+(\psi)). \end{aligned}$$

The aim of this section is to compute the behaviour of the tractor conformal superalgebra structure under these isomorphisms. As it turns out, the maps i_0 and i_1 allow us to identify our tractor conformal superalgebra with conformal superalgebras constructed for Lorentzian conformal spin manifolds in [Raj06] and [Hab96].

Proposition 6.9 *For fixed $g \in c$ it holds for all $\alpha, \beta \in \mathfrak{g}_0$ that*

$$i_0\left([\alpha,\beta]_{\mathfrak{g}_0}\right) = \left[V_\beta, V_\alpha\right]_{\mathfrak{X}(M)} = \left[i_0(\beta), i_0(\alpha)\right]_{\mathfrak{X}(M)}$$

Proof. We start with some algebraic computations: Assume that $\alpha, \beta \in \Lambda^2_{2,n}$. Wrt. the decomposition (1.11) we may write $\alpha = e^\flat_+ \wedge \alpha_+ + \alpha_0 + \alpha_\mp \cdot e^\flat_- \wedge e^\flat_+ + e^\flat_- \wedge \alpha_-$ with

$\alpha_+ = \sum_{i=1}^n \epsilon_i \alpha_i^+ \cdot e_i^\flat$, $\alpha_- = \sum_{i=1}^n \epsilon_i \alpha_i^- \cdot e_i^\flat$, $\alpha_0 = \sum_{i<j} \epsilon_i \epsilon_j \alpha_{ij}^0 \cdot e_i^\flat \wedge e_j^\flat$ for real coefficients α_i^+ etc. We let $E_{\pm,i} := \frac{1}{\sqrt{2}}(E_{n+1i} \pm E_{0i})$. Then the endomorphism $\alpha_E = \tau(\alpha) \in \mathfrak{so}(2,n)$ is given by

$$\alpha_E = \sum_{i=1}^n \epsilon_i \alpha_i^+ E_{+i} + \sum_{i=1}^n \epsilon_i \alpha_i^- E_{-i} + \alpha_\mp E_{n+10} + \sum_{i<j}^n \epsilon_i \epsilon_j \alpha_{ij}^0 E_{ij}.$$

An analogous expression holds for β_E. Using the algebra relations (1.1), it is straightforward to compute the following commutators for $i,j = 1,...,n$:

$$\begin{aligned}
[E_{\pm,i}, E_{\pm,j}] &= 0, \\
[E_{-,i}, E_{+,j}] &= E_{ij} - \epsilon_i \delta_{ij} E_{0n+1} + \epsilon_j \delta_{ij} E_{0n+1}, \\
[E_{\pm,i}, E_{n+10}] &= \mp E_{\pm,i}, \\
[E_{ij}, E_{\pm,k}] &= \epsilon_i \delta_{ik} E_{\pm,j} - \epsilon_j \delta_{jk} E_{\pm,i}.
\end{aligned}$$

With these formulas, we compute

$$\begin{aligned}
[\alpha_E, \beta_E]_{\mathfrak{so}(2,n)} = &+ \sum_{i=1}^n \epsilon_i (\beta_i^+ \alpha_\mp - \alpha_i^+ \beta_\mp) E_{+,i} + \sum_{i<j} \epsilon_i \epsilon_j (\alpha_{ij}^0 \beta_i^+ - \beta_{ij}^0 \alpha_i^+) E_{+,j} \\
&- \sum_{j<i} \epsilon_i \epsilon_j (\alpha_{ji}^0 \beta_j^+ - \beta_{ji}^0 \alpha_j^+) E_{+,i} + \text{Terms not involving } E_{+,i}.
\end{aligned}$$

A global version of this formula yields that for $\alpha, \beta \in \mathfrak{g}_0$ one has wrt. $g \in c$

$$proj_{\Lambda,+}^g ([\alpha,\beta]_{\mathfrak{g}_0}) = \alpha_\mp \cdot \beta_+ - \beta_\mp \alpha_+ + \sum_{i<j} \epsilon_i \epsilon_j (\alpha_{ij}^0 \beta_i^+ - \beta_{ij}^0 \alpha_i^+) s_j^\flat - \sum_{j<i} \epsilon_i \epsilon_j (\alpha_{ji}^0 \beta_j^+ - \beta_{ji}^0 \alpha_j^+) s_i^\flat,$$

where $(s_1, ..., s_n)$ is a local g–pseudo-orthonormal frame in TM, with coefficients of α taken with respect to this frame. This can be rewritten as

$$i_0 ([\alpha,\beta]_{\mathfrak{g}_0}) = \left(proj_{\Lambda,+}^g ([\alpha,\beta]_{\mathfrak{g}_0})\right)^\sharp = \alpha_\mp \cdot V_\beta - \beta_\mp V_\alpha + \left(V_\beta \lrcorner \alpha_0 - V_\alpha \lrcorner \beta_0\right)^\sharp \tag{6.9}$$

We now compare this expression to the Lie bracket $[V_\alpha, V_\beta]$. Dualizing the first nc-Killing equation (3.4) for α_+ yields that

$$\nabla_X^g V_\alpha = (X \lrcorner \alpha_0)^\sharp + \alpha_\mp \cdot X \;\; \forall X \in \mathfrak{X}(M).$$

Consequently,

$$[V_\beta, V_\alpha] = \nabla_{V_\beta}^g V_\alpha - \nabla_{V_\alpha}^g V_\beta = \left(\alpha_\mp V_\beta - \beta_\mp V_\alpha\right) + \left(V_\beta \lrcorner \alpha_0 - V_\alpha \lrcorner \beta_0\right)^\sharp. \tag{6.10}$$

Comparing the two expressions (6.9) and (6.10) immediately yields the claim. □

The next Proposition will be proved in a more general setting in Proposition 6.41:

Proposition 6.10 *For $\alpha \in \mathfrak{g}_0$, $\psi \in \mathfrak{g}_1$, and $g \in c$ such that $\varphi = \widetilde{\Phi}^g (proj_+^g \psi) = i_1(\psi)$, and $V_{\alpha_+} = i_0(\alpha)$, we have that*

$$i_1 \left([\alpha,\psi]_{\mathfrak{g}_1}\right) = \frac{1}{2} (\widetilde{\Phi}^g \circ proj_+^g) (\alpha \cdot \psi) = -\underbrace{\left(\nabla_{V_\alpha} \varphi + \frac{1}{4} \tau (\nabla V_\alpha) \cdot \varphi\right)}_{=: V_\alpha \circ \varphi},$$

where $\tau(\nabla V_\alpha) := \sum_{j=1}^n \epsilon_j \left(\nabla_{s_j} V_\alpha\right) \cdot s_j + (n-2) \cdot \lambda_{\alpha_+}$ and $L_{V_{\alpha_+}} g = 2 \lambda_{\alpha_+} g$.

Remark 6.11 The above term $V_\alpha \circ \varphi$ is the spinorial Lie derivative which is used in [Kos72, Hab96, Raj06] for the construction of a conformal Killing superalgebra.

Finally, we give the metric expression of the odd-odd bracket. Let $\psi \in \mathfrak{g}_1$ and $\varphi = \widetilde{\Phi}^g\left(proj^g_+(\psi)\right) = i_1(\psi)$:

$$i_1\left([\psi,\psi]\right) = \left(proj^g_{\Lambda,+}\left(\alpha^2_\psi\right)\right)^\sharp \overset{(3.7)}{=} c^1_{1,1}\cdot\left(\alpha^1_\varphi\right)^\sharp = c^1_{1,1}\cdot V_\varphi, \tag{6.11}$$

where the nonzero constant $c^1_{1,1} \in \mathbb{R}$ from (3.7) depends only on the choice of an admissible scalar product on $\Delta^{\mathbb{C}}_{2,n}$. These computations directly prove the following statement:

Theorem 6.12 *Given a Lorentzian conformal spin manifold $(M^{1,n-1},c)$, the associated tractor conformal superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = Par\left(\Lambda^2_{\mathcal{T}}(M),\nabla^{nc}\right) \oplus Par\left(\mathcal{S}(M),\nabla^{nc}\right)$ is via a fixed $g \in c$ isomorphic to the conformal superalgebra (6.2) on $\mathfrak{X}^{nc}(M) \oplus ker\ P^g$ (as considered in [Raj06]). Up to prefactors, the $g-$dependent maps i_0 and i_1 are superalgebra (anti-)isomorphisms.*

Remark 6.13 We defined the even part of the tractor conformal superalgebra to be (isomorphic to) the space of *normal* conformal vector fields. It is possible to include all conformal vector fields $\mathfrak{X}^c(M)$ in the even part using tractor calculus as follows: Let $\alpha \in \Omega^2_{\mathcal{T}}(M)$ be a tractor 2-form on (M,c) and let $V_\alpha = \left(proj^g_{\Lambda,+}(\alpha)\right)^\sharp \in \mathfrak{X}(M)$ be the associated vector field, which does not depend on the choice of $g \in c$. As proved in [Gov06, GS08], we have that $V_\alpha \in \mathfrak{X}^c(M)$ if and only if

$$\nabla^{nc}_X \alpha = \tau^{-1}\left(R^{\nabla^{nc},\mathcal{T}(M)}(V_\alpha, X)\right)\ \forall X \in \mathfrak{X}(M), \tag{6.12}$$

where we identify the skew-symmetric curvature endomorphism with a tractor 2-form by means of the isomorphism τ from (6.3). We now consider the extended tractor superalgebra

$$\mathfrak{g}^{ec}_0 := \{\alpha \in \Omega^2_{\mathcal{T}}(M) \mid \alpha \text{ satisfies (6.12)}\} \text{ and } \mathfrak{g}^{ec} := \mathfrak{g}^{ec}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_1 = Par(\mathcal{S},\nabla^{nc})$ is as before. On this space, we may define the same brackets as defined on $\mathfrak{g}$ in the previous section and observe that they are still well-defined: For $\alpha,\beta \in \mathfrak{g}^{ec}_0$, we have that also $[\alpha,\beta] \in \mathfrak{g}^{ec}_0$ as by Proposition 6.9 $V_{[\alpha,\beta]} = -[V_\alpha,V_\beta]_{\mathfrak{X}(M)}$, which is a conformal vector field. Next, let $\alpha \in \mathfrak{g}^{ec}_0$ and $\psi \in \mathfrak{g}_1$. Then we have that

$$\begin{aligned}\nabla^{nc}_X(\alpha\cdot\psi) &= (\nabla^{nc}_X\alpha)\cdot\psi = \tau^{-1}\left(R^{\nabla^{nc},\mathcal{T}(M)}(V_\alpha,X)\right)\cdot\psi \\ &\overset{\text{Prop. 3.41}}{=} 2\cdot R^{\nabla^{nc},\mathcal{S}}(V_\alpha,X)\psi \overset{\psi\in\mathfrak{g}_1}{=} 0,\end{aligned}$$

i.e. $\alpha\cdot\psi \in \mathfrak{g}_1$. This shows that $\mathfrak{g}^{ec}$ together with the defined brackets is a conformal superalgebra which naturally extends $\mathfrak{g}$. Moreover, Propositions 6.9 and 6.10 and (6.11) still hold in this situation and describe $\mathfrak{g}^{ec}$ wrt. a metric $g \in c$ as their proofs only involve the conformal Killing equation for vector fields and not the normalisation conditions.
However, we will only consider the superalgebra $\mathfrak{g}$ and not its extension $\mathfrak{g}^{ec}$ in the sequel because in the case of twistor spinors there are always normal conformal vector fields, and it seems to us that the structure of the subalgebra $\mathfrak{g}$ and the existence of distinguished normal conformal vector fields is more directly related to special geometric structures (cf. [Lei05]) on (M,c) than the structure of $\mathfrak{g}^{ec}$ as we will see in the next sections.

Remark 6.14 If for some fixed $g \in c$ the manifold admits geometric Killing spinors, for instance if there is an Einstein metric in the conformal class, the restrictions of the brackets (6.2) to $\mathfrak{X}^k(M) \oplus \mathcal{K}(M)$, the space of Killing vector fields and Killing spinors as even and odd parts, is well-defined (cf. [Far99]), and thus gives a subalgebra of the superalgebra $\mathfrak{g}^{ec}$.

6.3 Examples

The tractor conformal superalgebra in case of irreducible conformal holonomy

Let $(M^{1,n-1}, c)$ be a simply-connected Lorentzian conformal spin manifold of dimension ≥ 3. All Lorentzian conformal holonomy groups acting irreducible are known:

Proposition 6.15 ([SL11, ASL14]) *If $Hol(M,c)$ acts irreducibly on $\mathbb{R}^{2,n}$, then it is conjugate to either $SO^+(2,n)$ or $SU(1,\frac{n}{2})$ for even n.*

Note that in the first case the lift of $\mathfrak{hol}(M,c)$ to $\mathfrak{spin}(2,n)$ does not annihilate any non-trivial spinor. Whence the odd algebra $\mathfrak{g}_1$ is trivial in this case and no twistor spinors exist. We therefore restrict our attention to the second case, and thus assume that n is even:

Proposition 6.16 *For Lorentzian conformal structures with $Hol(M,c) = SU\left(1,\frac{n}{2}\right)$ one has that $dim_{\mathbb{C}}\mathfrak{g}_1 = 2$ and the tractor conformal algebra is **no** Lie superalgebra.*

Proof. In order to prove this Proposition, we start with the observation that by the discussion from section 3.2 complex parallel spin tractors on M correspond (after fixing a basepoint) to spinors in $\Delta^{\mathbb{C}}_{2,n}$ which are annihilated by the action of $\lambda_*^{-1}\left(\mathfrak{su}\left(1,\frac{n}{2}\right)\right)$. Let us call the space of these spinors $V_{\mathfrak{su}}$. We fix the following *complex* representation of the complex Clifford algebra $Cl^{\mathbb{C}}_{2,n}$ with $n+2 =: 2m$ on $\mathbb{C}^{2^m}$ (cf. [Bau81]): Let E, D, U and V denote the 2×2 matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ D = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Furthermore, let $\tau_j = \begin{cases} 1 & \epsilon_j = 1, \\ i & \epsilon_j = -1. \end{cases}$ $Cl^{\mathbb{C}}(p,q) \cong M_{2^m}(\mathbb{C})$ as complex algebras, and an explicit realisation of this isomorphism is given by

$$\Phi_{p,q}(e_{2j-1}) = \tau_{2j-1}\cdot E\otimes \ldots \otimes E \otimes U \otimes \underbrace{D\otimes \ldots \otimes D}_{(j-1)\times},$$
$$\Phi_{p,q}(e_{2j}) = \tau_{2j}\cdot E\otimes \ldots \otimes E \otimes V \otimes \underbrace{D\otimes \ldots \otimes D}_{(j-1)\times}.$$

We set $\widetilde{u}(\epsilon) := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 1 \\ -i\epsilon \end{pmatrix}$ for $\epsilon = \pm 1$ and introduce the spinors $\widetilde{u}(\epsilon_m, \ldots, \epsilon_1) := \widetilde{u}(\epsilon_m)\otimes \ldots \otimes \widetilde{u}(\epsilon_1)$ which form a basis of $\Delta^{\mathbb{C}}_{2,n}$. We work with the $Spin^+(2,n)$-invariant scalar product

$\langle u, v\rangle_{\Delta^{\mathbb{C}}_{2,n}} := i(e_1 \cdot e_2 \cdot u, v)_{\mathbb{C}^{2^m}}$. One calculates that

$$\langle \widetilde{u}(\epsilon_m, ..., \epsilon_1), \widetilde{u}(\delta_m, ..., \delta_1)\rangle = \begin{cases} 0 & (\epsilon_m, ..., \epsilon_1) \neq (\delta_m, ..., \delta_1) \\ \epsilon_1 & (\epsilon_m, ..., \epsilon_1) = (\delta_m, ..., \delta_1) \end{cases} \tag{6.13}$$

It is now straightforward to compute (cf. [Kat99]) that

$$V_{\mathfrak{su}} := \{v \in \Delta^{\mathbb{C}}_{2,n} \mid \lambda_*^{-1}\left(\mathfrak{su}\left(1, \frac{n}{2}\right)\right) \cdot v = 0\} = \mathrm{span}_{\mathbb{C}}\{u_+ := \widetilde{u}(1, ..., 1), u_- := \widetilde{u}(-1, ..., -1)\}. \tag{6.14}$$

Another straightforward computation involving (1.16) and (6.13) yields that

$$\alpha^2_{u_\pm} = \sum_{i=1}^{\frac{n}{2}+1} \epsilon_{2i} \cdot e^\flat_{2i-1} \wedge e^\flat_{2i}, \tag{6.15}$$

from which follows that

$$\alpha^2_{u_+} \cdot u_+ = i \cdot \left(\frac{n}{2} - 1\right) u_+ \neq 0. \tag{6.16}$$

If we turn to geometry, a global version of the previous observations shows that for simply-connected conformal structures with irreducible holonomy $SU(1, \frac{n}{2})$ the dimension of the complex space of twistor spinors is two-dimensional and (6.16) yields that the tractor conformal superalgebra is no Lie superalgebra as the odd-odd-odd Jacobi identity is not satisfied. □

Tractor conformal superalgebras with one twistor spinor

As a next special geometric situation we consider the case that dim $\mathfrak{g}_1 = 1$, i.e. there is only one linearly independent complex twistor spinor on (M, c). Such examples are easy to generate, as one might for example take a generic Lorentzian metric admitting a parallel spinor as classified in [Bry00] for low dimensions. In this context, the construction of a tractor conformal *Lie* superalgebra always works:

Proposition 6.17 *Suppose that the tractor conformal superalgebra of a simply-connected Lorentzian conformal spin manifold $(M^{1,n-1}, c)$ satisfies dim $\mathfrak{g}_1 = 1$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra.*

Proof. We fix a nontrivial twistor spinor $\psi \in \mathfrak{g}_1$ which is unique up to multiplication in $\mathbb{C}^*$ and assume that ker $\psi = \{0\}$. If this would be the case, Proposition 1.24 and Remark 1.25 imply that up to conjugation one has one of the cases

$$Hol(M, c) \subset \begin{cases} SU\left(1, \frac{n}{2}\right) & (3) \\ O(r) \times SU(1, \frac{n-r}{2}) & (4) \end{cases}$$

for some r, where the numbers correspond to the cases in Remark 1.25. As seen in the previous section, (3) implies that there exist at least two linearly independent twistor spinors. In case (4) we have that the representation ρ of $Hol(\overline{\mathcal{Q}}^1_+, \overline{\widetilde{\omega}}^{nc}) \subset Spin^+(2, n)$ on

$\Delta^{\mathbb{C}}_{2,n}$ splits into a product of representations $\rho \cong \rho_1 \otimes \rho_2$ on $Spin^+(0,r) \times Spin^+(2,n-r)$. Furthermore,

$$\Delta^{\mathbb{C}}_{2,n} \cong \Delta^{\mathbb{C}}_{0,r} \otimes \Delta^{\mathbb{C}}_{2,n-r},$$

considered as $Spin^+(0,r)\times Spin^+(2,n-r)$-representations. As there exists a $Hol(\overline{\mathcal{Q}}^1_+, \overline{\overline{\omega}}^{nc})$-invariant spinor in $\Delta^{\mathbb{C}}_{2,n}$, we conclude (cf. [Lei04]) that each of the factors ρ_1 and ρ_2 admits an invariant spinor. However, $(\rho_2)_*$ is the action of a subalgebra of $\mathfrak{su}(1,\frac{n-r}{2})$ on $\Delta^{\mathbb{C}}_{2,n-r}$ and as seen in the previous section, it annihilates at least two linearly independent complex spinors. Consequently, the representation ρ_2 fixes at least two linearly independent complex spinors and ρ_1 fixes at least one nontrivial complex spinor such that ρ fixes at least two linearly independent complex spinors which means that dim $\mathfrak{g}_1 > 1$, in contradiction to our assumption. Consequently, we have that ker $\psi \neq \{0\}$ for every $\psi \in \mathfrak{g}_1$. The second part of the proof of Theorem 6.5 then shows that $\mathfrak{g}$ is a Lie superalgebra. □

Corollary 6.18 *If the tractor conformal superalgebra associated to a simply-connected Lorentzian conformal spin manifold (M,c) is no Lie superalgebra, then there exist at least two linearly independent complex twistor spinors on (M,c).*

The tractor conformal superalgebra of flat Minkowski space

We describe the even part of the conformal algebra of flat Minkowski space $\mathbb{R}^{1,n-1}$ in terms of conformal tractor calculus and discuss extensions to a superalgebra. In physics notation (cf. [Sch08, Raj06]), the conformal algebra of Minkowski space $\mathbb{R}^{1,n-1}$ with coordinates x^i and the standard flat metric g_{ij} is generated by P_i, M_{ij}, D and K_i - corresponding to translations, rotations, the dilatation and the special orthogonal transformations:

$$\begin{aligned} P_i &= \partial_i, \\ M_{ij} &= x_i\partial_j - x_j\partial_i, \\ D &= x^i\partial_i, \\ K_i &= 2x_i x^j\partial_j - g(x,x)\partial_i. \end{aligned}$$

The Lie brackets are given by

$$\begin{aligned} [P_i, P_j] &= 0, \\ [M_{ij}, P_k] &= g_{jk}P_i - g_{ik}P_j, \\ [M_{ij}, M_{kl}] &= g_{jk}M_{il} - g_{ik}M_{jl} - g_{jl}M_{ik} + g_{il}M_{jk}, \\ [P_i, D] &= P_i, \\ [K_i, D] &= -K_i, \\ [P_i, K_j] &= 2g_{ij}D - 2M_{ij}, \\ [M_{ij}, K_k] &= g_{jk}K_i - g_{ik}K_j. \end{aligned}$$

As $\mathbb{R}^{1,n-1}$ is conformally flat, all conformal vector fields are automatically normal conformal, and thus the above vector fields generate the algebra $\mathfrak{X}^{nc}(\mathbb{R}^{1,n-1}) = \mathfrak{X}^c(\mathbb{R}^{1,n-1})$. We now consider the following natural isomorphism:

$$\tau_0 : \mathfrak{X}^{nc}(\mathbb{R}^{1,n-1}) \overset{g}{\cong} Par\left(\Lambda^2_{\mathcal{T}}\left(\mathbb{R}^{1,n-1}\right), \nabla^{nc}\right) \overset{\alpha\mapsto\alpha(0)}{\cong} \mathfrak{so}(2,n), \tag{6.17}$$

yielding that for flat Minkowski space $\mathfrak{g}_0 \cong \mathfrak{so}(2,n)$. As we shall see, the tractor approach establishes a link between the grading components of $\mathfrak{so}(2,n)$ from (2.13) and the various types of normal conformal vector fields:

For a translation $P_i = \partial_i$ the identification (6.17) works as follows: The dual 1-form is given by $\alpha_+ = P_i^\flat = \epsilon_i dx_i \in \Omega^1(\mathbb{R}^{1,n-1})$. Clearly, $\nabla^g \alpha_+ = 0$. The nc-Killing equations (3.4) with $K^g = 0$ then imply that $\alpha_\mp = 0, \alpha_0 = 0$ and $\alpha_- = 0$. It follows that the associated parallel tractor 2-form $\alpha^2_{P_i} \in Par\left(\Lambda^2_{\mathcal{T}}\left(\mathbb{R}^{1,n-1}\right)\right)$ satisfies $\alpha^2_{P_i}(0) = e^\flat_+ \wedge e^\flat_i$. Thus, given an arbitrary translation $P_i x^i$ for constant $x \in \mathbb{R}^{1,n-1}$, the associated skew-symmetric endomorphism is given by

$$\tau_0\left(P_i x^i\right) = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^\flat & 0 \end{pmatrix} \in \mathfrak{b}_{-1} \cong \mathbb{R}^n,$$

where $\mathfrak{b}_{-1}$ is the grading component of $\mathfrak{so}(2,n)$ from (2.13). Analogously, it is straightforward to compute:

$$\tau_0(a \cdot D) = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} \in \mathfrak{b}_0 \cong \mathfrak{co}(1,n-1),$$

$$\tau_0\left(M_{ij}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{b}_0 \cong \mathfrak{co}(1,n-1),$$

$$\tau_0\left(K^i y_i\right) = \begin{pmatrix} 0 & -y & 0 \\ 0 & 0 & y^\flat \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{b}_1 \cong \left(\mathbb{R}^n\right)^*,$$

where $a \in \mathbb{R}$ and $y \in (\mathbb{R}^n)^*$. Comparing the above brackets in $\mathfrak{X}^{nc}(\mathbb{R}^{1,n-1})$ and the brackets in $\mathfrak{so}(2,n)$ from section 2.4, one directly observes that τ_0 is actually a Lie algebra isomorphism. Thus, we now have a precise relationship between the various normal conformal vector fields and grading components of $\mathfrak{so}(2,n)$:

$$\begin{array}{ccccccccc} \mathfrak{so}(2,n) & \cong & \mathfrak{b}_{-1} & \oplus & \mathfrak{so}(p,q) & \oplus & \mathbb{R} & \oplus & \mathfrak{b}_1 \\ \tau_0 \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \mathfrak{X}^{nc}\left(\mathbb{R}^{1,n-1}\right) & \cong & \text{span } (P_i) & \oplus & \text{span } (M_{ij}) & \oplus & \text{span } (D) & \oplus & \text{span } (K_i) \\ & & \textit{translations} & & \textit{rotations} & & \textit{dilatation} & & \textit{sp. orth. transf.} \end{array}$$

Consequently, the isomorphism $\mathfrak{so}(2,n) \overset{\tau_0}{\cong} \mathfrak{X}^c\left(\mathbb{R}^{1,n-1}\right)$ yields a more geometric interpretation of the grading (2.4) of $\mathfrak{so}(2,n)$. We now consider the extension to a tractor conformal superalgebra: Solving the twistor equation on $\mathbb{R}^{1,n-1}$ is straightforward (cf. [BFGK91]): We have for a twistor spinor $\varphi \in \Gamma(\mathbb{R}^{1,n-1}, S^g_{\mathbb{C}}) \cong C^\infty\left(\mathbb{R}^{1,n-1}, \Delta^{\mathbb{C}}_{1,n-1}\right)$ using $K^g = 0$ that $\nabla D^g \varphi = 0$. Consequently, $D^g\varphi =: \varphi_1$ is a constant spinor. Integrating the twistor equation along the line $\{s \cdot x \mid 0 \le s \le 1\}$ yields that $\varphi(x) - \varphi(0) = -\frac{1}{n} x \cdot \varphi_1$. Thus, φ is of the form $\varphi(x) = \varphi_0 - \frac{1}{n} x \cdot \varphi_1$. Clearly, this establishes an isomorphism

$$\begin{array}{rccccccc} \tau_1: & \ker P^g & \to & \Delta^{\mathbb{C}}_{1,n-1} \oplus \Delta^{\mathbb{C}}_{1,n-1} & \cong & Par\left(\mathcal{S}(\mathbb{R}^{1,n-1}), \nabla^{nc}\right) & \cong & \Delta^{\mathbb{C}}_{2,n}, \\ & \varphi & \mapsto & \left(\varphi_0, -\frac{1}{n}\varphi_1\right) & \mapsto & \psi := \left(\widetilde{\Phi}^g\right)^{-1}\left(\varphi_0, -\frac{1}{n}\varphi_1\right) & \mapsto & \psi(0). \end{array}$$

Consequently, the tractor conformal superalgebra of $\mathbb{R}^{1,n-1}$is nothing but $\Lambda^2_{2,n} \oplus \Delta^{\mathbb{C}}_{2,n}$ with brackets as introduced in section 6.1. By means of τ_1 and τ_2 we have an identification

$$\mathfrak{g} \cong \Lambda^2_{2,n} \oplus \Delta^{\mathbb{C}}_{2,n} \overset{\tau_0,\tau_1}{\cong} \mathfrak{X}^{nc}(\mathbb{R}^{1,n-1}) \oplus \ker P^g, \tag{6.18}$$

and the right hand side of (6.18) is precisely the conformal superalgebra of Minkowski space wrt. the fixed standard metric as considered in [Raj06], for example. Again, one directly calculates that (6.18) is a superalgebra isomorphism. Thus, this example underlines that the tractor approach to conformal superalgebras is equivalent to the classical approaches. Using an explicit Clifford representation, one directly calculates that $\mathfrak{g}$ is no Lie superalgebra if $n > 3$, as also follows from Theorem 6.5. In case $n = 3$, and considering Minkowski space $\mathbb{R}^{2,1}$, there is a real structure on $\Delta^{\mathbb{C}}_{3,2}$, and restricting ourselves to real twistor spinors leads to the Lie superalgebra[3]

$$\mathfrak{X}^{nc}(\mathbb{R}^{2,1}) \oplus \ker P^g_{\mathbb{R}} \cong \Lambda^2_{3,2} \oplus \Delta^{\mathbb{R}}_{3,2} \subset \Lambda^2_{3,2} \oplus \Delta^{\mathbb{C}}_{3,2} = \mathfrak{X}^{nc}(\mathbb{R}^{2,1}) \oplus \ker P^g_{\mathbb{C}}.$$

Various Superalgebras for H^3

As a final example, we consider superalgebras defined over 3-dimensional hyperbolic space. Their structure reveals interesting relations to superalgebras similar to the Poincaré superalgebra. Superalgebras for H^3 arise in physics literature when studying supersymmetry of hyperbolic monopoles, as done for instance in [FG14].

H^3 equipped with its standard metric g is simply-connected, conformally flat and admits a 2-dimensional complex space $\mathcal{K}_{\pm}$ of Killing spinors to the Killing number $\pm\frac{i}{2}$. Their linear combinations span the 4-dimensional complex space of all twistor spinors $\ker P^g$. The conformal symmetries of H^3 are thus given by $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$\begin{aligned}\mathfrak{g}_0 &= Par(\Lambda^2_{\mathcal{T}}(H^3), \nabla^{nc}) \overset{g}{\cong} \mathfrak{X}^c(H^3) \cong \mathfrak{so}(1,4),\\ \mathfrak{g}_1 &= Par(\mathcal{S}(H^3), \nabla^{nc}) \overset{g}{\cong} \ker P^g(H^3) \cong \Delta^{\mathbb{C}}_{1,4}.\end{aligned}$$

We introduce a tractor superalgebra structure on this Riemannian space precisely as for the Lorentzian case, i.e. the brackets are given by

$$\begin{aligned}(\alpha,\beta) &\mapsto [\alpha,\beta],\\ (\alpha,\psi) &\mapsto \frac{1}{2}\alpha\cdot\psi,\\ (\psi_1,\psi_2) &\mapsto \alpha^2_{\psi_1,\psi_2},\end{aligned}$$

where $\alpha, \beta \in \mathfrak{so}(1,4) \cong \Lambda^2_{1,4}$, $\psi \in \Delta^{\mathbb{C}}_{1,4}$. This construction is nontrivial: A straightforward calculation reveals that in signature $(1,4)$ we have $\alpha^2_{\psi,\psi} = 0$ iff $\psi = 0$. One checks that with the so defined brackets all Jacobi identities are satisfied except the odd-odd-odd identity which by polarization is equivalent to

$$\alpha^2_{\psi}\cdot\psi = 0 \tag{6.19}$$

[3] The odd-odd-odd Jacobi identity holds in this case as every nonzero spinor $v \in \Delta^{\mathbb{R}}_{3,2}$ is pure, from which $\alpha^2_{v,v}\cdot v = 0$ follows. Note that there is no real structure on $\Delta^{\mathbb{R}}_{2,3}$.

for all $\psi \in \Delta^{\mathbb{C}}_{1,4}$. However, fixing an explicit realisation of Clifford multiplication shows that (6.19) holds iff $\langle \psi, \psi \rangle_{\Delta^{\mathbb{C}}_{1,4}} = 0$. Therefore, $\mathfrak{g}$ is *no* Lie superalgebra.

In the rest of this section we will show that $\mathfrak{g}$ contains an interesting nontrivial subalgebra which is a Lie superalgebra and we will then elaborate on how this algebra is realized in terms of spinor fields on H^3:
The odd part $\mathfrak{g}_1$ contains the subspace $\mathbb{R}^4 = \Delta^{\mathbb{R}}_{3,1} \subset \Delta^{\mathbb{C}}_{3,1} \subset \Delta^{\mathbb{C}}_{4,1} \cong \Delta^{\mathbb{C}}_{1,4}$, whereas the even part contains $\mathfrak{so}(3,1) = \mathfrak{so}(1,3) \subset \mathfrak{so}(1,4)$. Together they form $\overline{\mathfrak{g}} := \mathfrak{so}(3,1) \oplus \Delta^{\mathbb{R}}_{3,1}$. It is straightforward to calculate that the brackets of $\mathfrak{g}$ restrict to brackets on $\overline{\mathfrak{g}}$, i.e. $\overline{\mathfrak{g}}$ is a subalgebra of $\mathfrak{g}$ (in the *super* sense). Fixing an explicit spinor representation on $\Delta^{\mathbb{R}}_{3,1}$ reveals that $\alpha^2_{\psi} \cdot \psi = 0$ for all $\psi \in \Delta^{\mathbb{R}}_{3,1}$, i.e. $\overline{\mathfrak{g}}$ is a Lie superalgebra.

$\overline{\mathfrak{g}}$ is closely related to the super Poincaré algebra $\mathfrak{g}_p$, defined as follows: As a $\mathbb{Z}_2$-graded vector space, $\mathfrak{g}_p = (\mathfrak{so}(3,1) \ltimes \mathbb{R}^{3,1}) \oplus \Delta^{\mathbb{R}}_{3,1}$. The even-even bracket is the standard bracket on $\mathfrak{so}(3,1) \ltimes \mathbb{R}^{3,1}$, the even-odd-bracket is given by trivial extension of $(\alpha, \psi) \mapsto \frac{1}{2}\alpha \cdot \psi$, where $\alpha \in \mathfrak{so}(3,1)$ and $\psi \in \Delta^{\mathbb{R}}_{3,1}$, i.e. vectors act trivially on spinors, and the odd-odd bracket is given by squaring spinors to vectors, i.e. $(\psi_1, \psi_2) \mapsto V_{\psi_1,\psi_2} \in \mathbb{R}^{3,1}$. The relation to our Lie superalgebra $\overline{\mathfrak{g}}$ can be understood as follows: Consider the symmetrized tensor product $S^2\Delta^{\mathbb{R}}_{3,1} \subset \Delta^{\mathbb{R}}_{3,1} \otimes \Delta^{\mathbb{R}}_{3,1}$. There is an isomorphism

$$\begin{aligned} S^2\Delta^{\mathbb{R}}_{3,1} &\to \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1}, \\ \psi \otimes \psi &\mapsto (\alpha^2_{\psi}, V_{\psi}). \end{aligned} \tag{6.20}$$

Thus, the odd-odd brackets in $\mathfrak{g}_p$ and $\overline{\mathfrak{g}}$ correspond to projection of (6.20) to the first resp. second factor. Moreover, in $\overline{\mathfrak{g}}$ we omit the translation part $\mathbb{R}^{3,1}$ in the even part.

The super Poincaré algebra is geometrically realized on $\mathbb{R}^{3,1}$ by Killing vector fields ($\mathfrak{so}(3,1)$), parallel vector fields ($\mathbb{R}^{3,1}$) and parallel spinors ($\Delta^{\mathbb{R}}_{3,1}$). In contrast, a natural geometric realization of the Lie superalgebra $\overline{\mathfrak{g}}$ on H^3 is given as follows:
The complex spinor module $\Delta^{\mathbb{C}}_3 := \Delta^{\mathbb{C}}_{0,3} \cong \mathbb{C}^2$ admits a quaternionic structure which commutes with Clifford multiplication, i.e. there is a map $\gamma : \Delta^{\mathbb{C}}_3 \to \Delta^{\mathbb{C}}_3$ satisfying

$$\begin{aligned} \gamma^2 &= 1, \\ \gamma(iv) &= -i\gamma(v), \\ \gamma(x \cdot v) &= x \cdot \gamma(v), \end{aligned} \tag{6.21}$$

where $v \in \Delta^{\mathbb{C}}_3$ and $x \in \mathbb{R}^3$. We fix such a structure γ. Clearly, this induces a map $\gamma : S^g(H^3) \to S^g(H^3)$, which is a quaternionic structure commuting with Clifford multiplication in each fibre. Let $\varphi \in \mathcal{K}_+$ be a Killing spinor to Killing number $\frac{i}{2}$. (6.21) yields that $\gamma(\varphi) \in \mathcal{K}_-$. We consider the space $\Delta(\mathcal{K}_+ \oplus \mathcal{K}_-) := \{\varphi + \gamma(\varphi) \mid \varphi \in \mathcal{K}_+\}$. This is a 4-dimensional *real* subspace of ker $P^g(H^3)$. It is not closed under multiplication with i.
Consider furthermore $\mathfrak{X}^k(H^3) \cong \mathfrak{so}(1,3)$, the space of Killing vector fields on (H^3, g). Combining with the above introduced spinor fields we obtain the space

$$\mathfrak{X}^k(H^3) \oplus \Delta(\mathcal{K}_+ \oplus \mathcal{K}_-). \tag{6.22}$$

It is easy to verify that (6.22) is a subalgebra of $\mathfrak{g} = \mathfrak{X}^c(H^3) \oplus \ker P^g(H^3) = \mathfrak{so}(1,4) \oplus \Delta^{\mathbb{C}}_{1,4}$. Moreover, as Killing spinors are determined by their value at a point, it is straightforward to calculate that (6.22) equipped with this superalgebra structure is isomorphic to the Lie superalgebra $\overline{\mathfrak{g}} \subset \mathfrak{g}$.

To summarize: The tractor superalgebra $\mathfrak{g}$ of conformal symmetries of H^3 is no Lie superalgebra. It contains a Lie subalgebra $\overline{\mathfrak{g}}$ which is conceptually very similar to the Poincaré superalgebra $\overline{\mathfrak{g}}_p$ of $\mathbb{R}^{3,1}$. Geometrically, $\overline{\mathfrak{g}}$ corresponds to Killing vector fields in the even part and special combinations of Killing spinors in the odd part.

6.4 A tractor conformal superalgebra with R-symmetries

We have shown that our tractor construction applied to geometries with special unitary conformal holonomy never leads to conformal Lie superalgebras. On the other hand, [MH13] describes a way of constructing a conformal Lie superalgebra for Fefferman spin spaces. It is shown that this is only possible under the inclusion of a further nontrivial conformal symmetry in the even part $\mathfrak{g}_0$ of the algebra. This symmetry has also been considered in physics and is known as R-symmetry. As observed in [Lei07, BJ10], having a Fefferman metric in the conformal class of an even-dimensional conformal structure is at least locally equivalent to $Hol(M,c) \subset SU\left(1,\frac{n}{2}\right)$. Consequently, we can view the construction in [MH13] as a possible way to overcome the problems in constructing a tractor conformal Lie superalgebra for special unitary holonomy which we faced in section 6.3. Our aim is therefore to reproduce the construction of conformal Lie superalgebras with R-symmetries for Fefferman spaces in the framework of the conformal tractor calculus.
Let (M,c) be an simply-connected, even-dimensional Lorentzian conformal spin manifold and $g \in c$. For the definition and construction of Fefferman spin spaces we refer to [Bau99, BL04, BJ10] or chapter 7. The following is a standard fact:

Proposition 6.19 ([Lei01, Bau99]) *On a Lorentzian Fefferman spin space $(M^{1,n-1},g)$ there are distinguished, linearly independent complex twistor spinors φ_ϵ for $\epsilon = \pm 1$ such that*

1. *The Dirac current V_{φ_ϵ} is a regular lightlike Killing vector field.*
2. *$\nabla_{V_{\varphi_\epsilon}} \varphi_\epsilon = ic\varphi_\epsilon$ for some $c \in \mathbb{R}\backslash\{0\}$.*

We now restrict ourselves to *generic* Fefferman spin spaces, i.e. our overall assumption in this section in terms of conformal data is

$$Hol(M^{1,n-1},c) \subset SU\left(1,\frac{n}{2}\right) \text{ and } \dim_{\mathbb{C}} \ker P^g = 2.$$

Note that in case $Hol(M^{1,n-1},c=[g]) = SU\left(1,\frac{n}{2}\right)$, the second requirement follows automatically as seen in section 6.3. Under these assumptions, we present a construction

$$\boxed{(M,c) \to \widetilde{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \text{ Lie superalgebra,}}$$

where $\widetilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \mathbb{R}$ is the algebra of parallel tractor 2-forms, extended via an additional symmetry (which will be made precise). The construction turns out to be canonical, i.e. it does not involve other choices.

Algebraic preparation

We want to investigate the space of parallel spin tractors on (M, c) more closely. To this end, we use the complex spinor representation on $\Delta^{\mathbb{C}}_{2,n}$ from section 6.3 and let $u_\pm$ be the spinors from (6.14). Let $W := \mathrm{span}_{\mathbb{C}}\{u_+, u_-\}$. We have already computed $\omega_0 := \alpha^2_{u_\pm, u_\pm}$ in (6.15). A straightforward, purely algebraic calculation reveals the following:

Proposition 6.20 *The pseudo-Kähler form ω_0 on $\mathbb{R}^{2,n}$ is distinguished by the following properties:*

1. *For every $w \in W$ there exists a constant $c_w \geq 0$ such that $\alpha^2_{w,w} = c_w \cdot \omega_0$.*
2. $\|\omega_0\|^2_{2,n} = \frac{n}{2} + 1$

Moreover, one calculates that for all $a, b \in \mathbb{C}$ we have $\frac{1}{i}\omega_0 \cdot (au_+ + bu_-) = \left(\frac{n}{2} - 1\right) \cdot (au_+ - bu_-)$, whence

$$\mathrm{span}_{\mathbb{C}}\{u_\pm\} = \mathrm{Eig}_{\mathbb{C}}\left(\frac{1}{i}\omega_0, \pm\left(\frac{n}{2} - 1\right)\right). \tag{6.23}$$

Lemma 6.21 *Consider $u_+ \in W$ and let $\alpha \in \Lambda^2_{2,n}$ be a 2-form. If $\alpha \cdot u_+ \in W$, then α can be written as $\alpha = \sum_{i=1}^{\frac{n+2}{2}} a_i \cdot e^\flat_{2i-1} \wedge e^\flat_{2i}$ for $a_i \in \mathbb{R}$. We denote the space of all these forms by V.*

Proof. We write a generic 2-form as $\alpha = \sum_{i<j} a_{ij} e^\flat_i \wedge e^\flat_j$. It follows that $\alpha \cdot u_\pm = \sum_{i<j} a_{ij} e_i \cdot e_j \cdot u_\pm$. Using our concrete realisation of Clifford multiplication, one calculates that $j \neq i + 1 \Rightarrow e_i e_j u_+ \propto u(1, ..., 1, -1, 1..., 1, -1..., 1)$, where -1 occurs at positions $\left\lfloor \frac{i+1}{2} \right\rfloor$ and $\left\lfloor \frac{j+1}{2} \right\rfloor$. As $\alpha \cdot u_+ \in W$, it follows that $a_{ij} = 0$ for these choices of i and j. □

Another purely algebraic computation reveals the following:

Lemma 6.22 *On W there exists a up to sign unique $\mathbb{C}$-linear map $\iota : W \to W$ such that $\iota^2 = 1$ and ι is an anti-isometry of $(W, \langle \cdot, \cdot \rangle_{\Delta^{\mathbb{C}}_{2,n}})$, i.e. $\langle \iota(u), \iota(v) \rangle_{\Delta^{\mathbb{C}}_{2,n}} = -\langle u, v \rangle_{\Delta^{\mathbb{C}}_{2,n}}$.*

Moreover, (6.23) shows that setting

$$\frac{1}{i}\omega_0 \cdot u =: \left(\frac{n}{2} - 1\right) \cdot l(u), \text{ for } u \in W \tag{6.24}$$

defines a unique $\mathbb{C}$–linear map $l : W \to W$. l is an isometry wrt. $\langle \cdot, \cdot \rangle_{\Delta^{\mathbb{C}}_{2,n}}$ and $l^2 = 1$. We note that wrt. the basis (u_+, u_-) of W, ι and l are given by $\iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $l = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. One easily calculates that for all $\alpha \in V$ and $u \in W$ we have

$$\begin{aligned} &\alpha \cdot \iota(u) = -\iota(\alpha \cdot u), \ \alpha \cdot l(u) = l(\alpha \cdot u), \\ &\iota(l(u)) = -l(\iota(u)). \end{aligned} \tag{6.25}$$

Geometric construction

We now turn to geometry again: Let (M, c) be a simply-connected Lorentzian conformal spin manifold with special unitary conformal holonomy and suppose that $\dim_{\mathbb{C}} W = 2$, where now $W = Par(\mathcal{S}_{\mathbb{C}}(M), \nabla^{nc})$. Global versions of our previous algebraic observations show: There exists a unique parallel tractor 2-form $\omega_0 \in Par\left(\Lambda^2_{\mathcal{T}}(M), \nabla^{nc}\right)$ distinguished by properties of Proposition 6.20. Furthermore, Clifford multiplication with $\frac{1}{i}\omega_0$ is an automorphism of W with eigenvalues $\pm(\frac{n}{2} - 1)$. We now fix $\psi_\pm \in \mathrm{Eig}\left(\frac{1}{i}\omega_0, \pm\left(\frac{n}{2} - 1\right)\right) \cap W$ with $\langle \psi_\pm, \psi_\pm \rangle_{\Delta^{\mathbb{C}}_{2,n}} = \pm 1$ and $\langle \psi_\pm, \psi_\mp \rangle_{\Delta^{\mathbb{C}}_{2,n}} = 0$. With these requirements, $\psi_\pm$ are unique up to multiplications with elements of $S^1 \subset \mathbb{C}$. In fact, if one fixes a Fefferman metric g in the conformal class, then $\widetilde{\Phi}^g(proj^g_+ \psi_\pm) = \varphi_\pm \in \ker P^g$(up to constant multiples), where $\varphi_\pm$ were introduced in Proposition 6.19. We further require that $\iota(\psi_+) = \psi_-$ which reduces the ambiguity in choosing $\psi_\pm$ to only one complex phase. We set

$$\mathfrak{g}_1 := W = \mathrm{span}_{\mathbb{C}}\{\psi_+, \psi_-\} \subset \mathcal{S}_{\mathbb{C}}(M).$$

On $\mathfrak{g}_1$ there are natural maps $\iota : \mathfrak{g}_1 \to \mathfrak{g}_1$ and $l : \mathfrak{g}_1 \to \mathfrak{g}_1$ with the same properties as the corresponding maps from the algebraic preparations. $\mathfrak{g}_1$ defines the odd part of the tractor superconformal algebra we are about to construct. For the construction of the even part, we first set as in section 6.1 $\mathfrak{g}_0 := Par\left(\Lambda^2_{\mathcal{T}}(M), \nabla^{nc}\right)$ and equip it with the bracket of endomorphisms.

Proposition 6.23 *For $Hol(M, c) \subset SU\left(1, \frac{n}{2}\right)$ and dim $\mathfrak{g}_1 = 2$, we have that $\mathfrak{g}_0$ is abelian and dim $\mathfrak{g}_0 \leq \frac{n}{2} + 1$.*

Proof. For $\alpha \in \mathfrak{g}_0$ and $\psi = \psi_+ \in \mathfrak{g}_1$ we must by the derivation property of ∇^{nc} wrt. Clifford multiplication have that also $\alpha \cdot \psi$ is a parallel spin tractor, i.e. $\alpha \cdot \psi \overset{!}{\in} \mathfrak{g}_1$. Lemma 6.21 then determines all possible forms of α and from this the statement is immediate. □

We now set $\widetilde{\mathfrak{g}}_0 := \mathfrak{g}_0 \oplus \mathbb{R}$, where the sum is a direct sum of abelian Lie algebras and thus $\widetilde{\mathfrak{g}}_0$ is abelian too. We introduce further brackets on $\mathfrak{g} := \widetilde{\mathfrak{g}}_0 \oplus \mathfrak{g}_1$:

$$\begin{aligned} &[\cdot,\cdot] : \widetilde{\mathfrak{g}}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1, \\ &\quad ((\alpha, a), \psi) \mapsto \frac{1}{i} \cdot (\alpha \cdot (\iota(\psi))) + a \cdot \iota(l(\psi)), \\ &[\cdot,\cdot] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \widetilde{\mathfrak{g}}_0, \\ &\quad (\psi_1, \psi_2) \mapsto \left(\alpha^2_{\psi_1,\psi_2}, \left(\frac{n}{2} - 1\right) \cdot \mathrm{Re}\left(\langle \psi_1, l(\psi_2) \rangle_{\mathcal{S}}\right)\right). \end{aligned} \tag{6.26}$$

Clearly, these brackets have the right symmetry properties to turn $\widetilde{\mathfrak{g}}_0 \oplus \mathfrak{g}_1$ into a superalgebra.

Theorem 6.24 *The superalgebra $\widetilde{\mathfrak{g}}_0 \oplus \mathfrak{g}_1$ associated to (M, c) canonically up to sign[4] is a Lie superalgebra.*

[4] In fact, defining the above brackets via the abstract maps ι and l rather than using the basis $\psi_\pm$ reveals that the construction involves no further choices once $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are determined.

Proof. All we have to do is checking the Jacobi identities: By polarization, the odd-odd-odd Jacobi identity is equivalent to $[[\psi,\psi]\psi] = 0$ for all $\psi \in \mathfrak{g}_1$. By definition, we have for $\psi = a\psi_+ + b\psi_-$ with $a, b \in \mathbb{C}$ that

$$\begin{aligned}
[[\psi,\psi],\psi] &= \left[\left(\alpha^2_{\psi,\psi}, \left(\frac{n}{2}-1\right)\cdot \mathrm{Re}\left(\langle \psi, l(\psi)\rangle_{\mathcal{S}}\right)\right), \psi\right] \\
&= \frac{1}{i}\cdot \alpha^2_{\psi,\psi}\cdot \iota(\psi) + \left(\frac{n}{2}-1\right)\cdot \mathrm{Re}\left(\langle \psi, l(\psi)\rangle_{\mathcal{S}}\right)\cdot \iota(l(\psi)) \\
&= \frac{1}{i}\left(|a|^2\omega_0 + |b|^2\omega_0\right)\cdot (a\psi_- + b\psi_+) + \left(\frac{n}{2}-1\right)(|a|^2+|b|^2)\cdot (a\psi_- - b\psi_+) \\
&= (|a|^2+|b|^2)\cdot \left(\frac{n}{2}-1\right)\cdot (-a\psi_- + b\psi_+) + \left(\frac{n}{2}-1\right)(|a|^2+|b|^2)(a\psi_- - b\psi_+) \\
&= 0.
\end{aligned}$$

As $\widetilde{\mathfrak{g}}_0$ is abelian, the even-odd-odd identity is by polarization equivalent to $[[(\alpha,\gamma),\psi],\psi] = 0$ for all $\alpha \in \mathfrak{g}_0$, $\gamma \in \mathbb{R}$ and $\psi \in \mathfrak{g}_1$. By definition of the brackets involved, this is the case iff

$$\left(\alpha^2_{\frac{1}{i}\cdot(\alpha\cdot\iota(\psi))+\gamma\cdot(\iota(l(\psi)))),\psi}, \left(\frac{n}{2}-1\right)\cdot \mathrm{Re}\left(\langle \frac{1}{i}\cdot(\alpha\cdot\iota(\psi)) + \gamma\cdot(\iota(l(\psi))), l(\psi)\rangle_{\mathcal{S}}\right)\right) \overset{!}{=} 0 \in \mathfrak{g}_0 \oplus \mathbb{R}. \tag{6.27}$$

We again write $\psi = a\psi_+ + b\psi_-$ for complex constants a and b. Lemma 6.21 implies that $\frac{1}{i}\alpha\cdot\psi_+ = d\cdot\psi_+$ for some real constant d and $\frac{1}{i}\alpha\cdot\psi_- = -d\cdot\psi_-$. Then the $\mathfrak{g}_0$– part of (6.27) is given by

$$\alpha^2_{-ad\psi_-+bd\psi_++\gamma(a\psi_--b\psi_+),a\psi_++b\psi_-} = 0,$$

where we used that $\alpha^2_{\psi_+,\psi_-} = 0$, and the $\mathbb{R}$–part of (6.27) is proportional to

$$\begin{aligned}
&\mathrm{Re}\left(\langle \frac{1}{i}\cdot(\alpha\cdot\iota(\psi)) + \gamma\cdot(\iota(l(\psi))), l(\psi)\rangle_{\mathcal{S}}\right) \\
=&\mathrm{Re}\left(\langle d\cdot(-a\cdot\psi_- + b\cdot\psi_+) + \gamma\cdot(a\psi_- - b\psi_+), a\psi_+ - b\psi_-\rangle_{\mathcal{S}}\right) \\
=&0.
\end{aligned}$$

Finally, since $\widetilde{\mathfrak{g}}_0$ is abelian, the even-even-odd identity is equivalent to

$$\left[(\alpha,a), \frac{1}{i}\beta\cdot(\iota(\psi)) + b\cdot\iota(l(\psi))\right] \overset{!}{=} \left[(\beta,b), \frac{1}{i}\alpha\cdot(\iota(\psi)) + a\cdot\iota(l(\psi))\right] \in \mathfrak{g}_1,$$

where $(\alpha,a),(\beta,b) \in \widetilde{\mathfrak{g}}_0$ and $\psi \in \mathfrak{g}_1$. Unwinding the definitions and using (6.25), we find that the left hand side is given by

$$\begin{aligned}
&\frac{1}{i}\left(\frac{1}{i}\alpha\cdot\iota(\beta\cdot\iota(\psi)) + a\cdot\iota(l(\beta\cdot\iota(\psi))) + b\cdot\alpha\cdot l(\psi)\right) + ab\cdot\iota(l(\iota(l(\psi)))) \\
=&\alpha\cdot\beta\cdot\psi + \frac{a}{i}\beta\cdot l(\psi) + \frac{b}{i}\cdot\alpha\cdot l(\psi) - ab\cdot\psi \overset{[\alpha,\beta]=0}{=} \beta\cdot\alpha\cdot\psi + \frac{a}{i}\beta\cdot l(\psi) + \frac{b}{i}\cdot\alpha\cdot l(\psi) - ab\cdot\psi \\
=&\left[(\beta,b), \frac{1}{i}\alpha\cdot(\iota(\psi)) + a\cdot\iota(l(\psi))\right].
\end{aligned}$$

These calculations prove the Theorem. □

Remark 6.25 Let $g \in c$ be a Fefferman metric on M. By means of g we identify the parallel spin tractors ψ_ϵ with the distinguished twistor spinors φ_ϵ from Proposition 6.19 for $\epsilon = \pm 1$ and parallel 2-form tractors with normal conformal vector fields. Calculations completely analogous to that in section 6.2 reveal that the even-odd bracket (6.26) is under this g-metric identification given by (extension of)

$$(\mathfrak{X}^{nc}(M) \oplus \mathbb{R}) \times \ker P^g \to \ker P^g,$$
$$((V,a),\varphi_\epsilon) \mapsto L_V \varphi_{-\epsilon} + \epsilon \cdot a \cdot \varphi_{-\epsilon},$$

and in this picture the odd-odd-odd Jacobi identity for $\widetilde{\mathfrak{g}}_0 \oplus \mathfrak{g}_1$ is equivalent to the existence of a constant ρ such that $L_{V_{\varphi_\epsilon}} \varphi_\epsilon + \epsilon \cdot \rho \cdot \varphi_\epsilon = 0$, as proved independently in [MH13].

Remark 6.26 There is an odd-dimensional analogue of this construction: Namely, consider the case of a simply-connected, Lorentzian Einstein-Sasaki manifold $(M^{1,n-1}, g)$ of negative scalar curvature (cf. [Lei01, Boh99]), which can be equivalently characterized in terms of special unitary holonomy of the cone over (M,g). It follows that (M,g) is spin and there again exist two distinguished conformal Killing spinors (cf. [BL04, Lei07]). Let us assume that the complex span of these twistor spinors is already $\ker P^g =: \mathfrak{g}_1$. As (M,g) is Einstein with $\text{scal}^g < 0$ there exists in this case a *distinguished* spacelike, parallel standard tractor τ, defining a holonomy reduction $Hol(M,[g]) \subset SU\left(1, \frac{n-1}{2}\right) \subset SO(2,n-1) \subset SO(2,n)$ and a splitting $\mathcal{T}(M) = \langle \tau \rangle^\perp \oplus \langle \tau \rangle$. Furthermore $\Delta_{2,n-1} \cong \Delta_{2,n}$ as $Spin(2,n-1)$-representations. Setting $\mathfrak{g}_0 := Par\left(\Lambda^2_{\mathcal{T}}(M), \nabla^{nc}\right) \cap \{\alpha \in \Omega^2_{\mathcal{T}}(M) \mid \alpha(\tau,\cdot) = 0\}$, we can then proceed completely analogous to the even-dimensional case just discussed, i.e. we perform in the tractor setting the same purely algebraic construction on the orthogonal complement of τ in $\mathcal{T}(M)$. This turns $(\mathfrak{g}_0 \oplus \mathbb{R}) \oplus \mathfrak{g}_1$ into a Lie superalgebra with R-symmetries. Again, the overall construction is canonical. For a construction which uses a fixed metric in the conformal class, we refer to [MH13].

6.5 Summary and application in small dimensions

We want to summarize the various possibilities and obstructions one faces in the attempt of constructing a conformal Lie superalgebra via the tractor approach in Lorentzian signature. To this end, recall that twistor spinors on Lorentzian manifolds can be categorized into three types according to Theorem 3.32: We have shown:

- If all twistor spinors are of type 1. or 2., the tractor conformal superalgebra is a Lie superalgebra (cf. Theorem 6.5). Moreover, if $\mathfrak{g}$ is a Lie superalgebra, there is a Brinkmann metric in the conformal class or a local splitting $[g] = [-dt^2 + h]$, where h is Riemannian Ricci-flat Kähler.
- The previous situation always occurs if the space of twistor spinors is 1-dimensional.
- If there are exactly two linearly independent twistor spinors of type $3.a$ or $3.b$ (depending on the dimension to be even or odd), one can construct a Lie superalgebra under the inclusion of an R-symmetry. Depending on the dimension, one has a Fefferman metric or a Lorentzian Einstein-Sasaki metric in the conformal class.

Remark 6.27 We have not yet discussed the case when the twistor spinor is of type $3.c$ in Theorem 3.32, i.e. when there is -at least locally - a splitting $(M,g) \cong (M_1,g_1) \times$

(M_2, g_2) into a product of Einstein spaces. By Theorem 3.17 we have that $Hol(M,[g]) \cong Hol(M_1,[g_1]) \times Hol(M_2,[g_2])$. In this situation, it is an algebraic fact (cf. [Lei04]) that every spinor $v \in \Delta_{2,n}$ which is fixed by $Hol(M,[g])$ is of the form $v = v_1 \otimes v_2$ where $Hol(M_i,[g_i])v_i = v_i$. As also the converse is trivially true, we see that on the level of tractor conformal superalgebras, the product case manifests itself in a splitting of the odd part of $\mathfrak{g}$, i.e. $\mathfrak{g}_1 = \mathfrak{g}_1^1 \otimes \mathfrak{g}_1^2$, where $\mathfrak{g}_1^i$ are the odd parts of the tractor conformal Lie superalgebras $\mathfrak{g}^i = \mathfrak{g}_0^i \oplus \mathfrak{g}_1^i$ of $(M_i,[g_i])$ for $i = 1,2$. Moreover, note that we *never* have a splitting in the even part, $\mathfrak{g}_0 \neq \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$. This is because as (M_i, g_i) are Einstein manifolds, there are parallel standard tractors $t_i \in \mathcal{T}(M_i)$, and it follows that $t_1 \wedge t_2 \in \mathfrak{g}_0$, but $t_1 \wedge t_2 \notin \mathfrak{g}_0 \oplus \mathfrak{g}_1$. It is moreover clear from the structure of α_ψ^2 from Theorem 3.32 in this situation that $\mathfrak{g}$ is no Lie superalgebra in this case. [MH13] presents a way of extending $\mathfrak{g}$ to a Lie superalgebra under the inclusion of R-symmetries.

We have now studied the construction of a tractor conformal superalgebra for every (local) geometry admitting twistor spinors and summarize our results:

Theorem 6.28 *Let $(M^{1,n-1}, c)$ be a Lorentzian conformal spin manifold admitting twistor spinors. Assume further that all twistor spinors on (M,c) are of the same type according to Theorem 3.32. Then there are the following relations between special Lorentzian geometries in the conformal class c and properties of the tractor conformal superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of (M,c):*

Twistor spinor type (Thm. 3.32)	*Special geometry in c*	*Structure of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$*
1.	*Brinkmann space*	*Lie superalgebra*
2.	*Splitting $(\mathbb{R}, -dt^2)\times$ Riem. Ricci-flat*	*Lie superalgebra*
3.a	*Lorentzian Einstein Sasaki (n odd)*	*No Lie superalgebra, becomes Lie superalgebra under inclusion of nontrivial R-symmetry*
3.b	*Fefferman space (n even)*	*No Lie superalgebra, becomes Lie superalgebra under inclusion of nontrivial R-symmetry*
3.c	*Splitting $M_1 \times M_2$ into Einstein spaces*	*No Lie superalgebra, odd part splits $\mathfrak{g}^i = \mathfrak{g}_0^i \otimes \mathfrak{g}_1^i$, but $\mathfrak{g}_0 \neq \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$*

Let us apply this statement to tractor conformal superalgebras $\mathfrak{g}$ of non conformally flat Lorentzian conformal manifolds $(M^{1,n-1},[g])$ admitting twistor spinors in small dimensions which have been studied in [Lei01, BL04]:

Let n=3. It is known that dim ker $P^g \leq 1$ in this situation. Consequently, by Proposition 6.17 $\mathfrak{g}$ is a tractor conformal Lie superalgebra. Every twistor spinor is off a singular set locally equivalent to a parallel spinor on a *pp*-wave.
Let n=4. Here, dim ker $P^g \leq 2$. Exactly one of the following cases occurs: Either, there is a Fefferman metric in the conformal class with two linearly independent twistor spinors. In this case we can construct a tractor superalgebra with R-symmetries. Otherwise, all twistor spinors are locally equivalent to parallel spinors on pp-waves. In this case the

ordinary construction of a tractor conformal *Lie* superalgebra $\mathfrak{g}$ works.
Let n=5. This case is already more involved but the possibility of constructing a tractor conformal Lie superalgebra can be completely described: One again has that dim ker $P^g \leq 2$. Exactly one of the following cases occurs:

1. There is a Lorentzian Einstein Sasaki metric in the conformal class. In this case, dim ker $P^g = 2$ and one can construct a tractor conformal Lie superalgebra with R-symmetries as indicated in Remark 6.26.

2. (M,g) is (at least locally) conformally equivalent to a product $\mathbb{R}^{1,0} \times (N^4, h)$, where the last factor is Riemannian Ricci-flat Kähler and admits two linearly independent parallel spinors. This corresponds to type 2. twistor spinors from Theorem 3.32, and thus one can construct a tractor conformal Lie superalgebra.

3. All twistor spinors are equivalent to parallel spinors on *pp*-waves. Again, the construction yields a Lie superalgebra.

Let $n \geq 6$. Now *mixtures* can occur, i.e. it is possible that some twistor spinors are of type 1. or 2., and some twistor spinors are of type 3. according to Theorem 3.32. In this case, Theorem 6.28 does not apply.

6.6 Construction of tractor conformal superalgebras for higher signatures

Let $(M^{p,q}, c)$ be a conformal spin manifold of *arbitrary* signature (p,q) with $p+q=n$ and complex space of parallel spin tractors $\mathfrak{g}_1 = Par(\mathcal{S}_{\mathbb{C}}(M), \nabla^{nc})$. We want to associate to (M,c) a tractor conformal superalgebra in a natural way. However, our construction from the previous sections depends crucially on Lorentzian signature. More precisely, the bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ may become trivial in other signatures (cf. Remark 1.23), and it therefore has to be modified: As remarked in section 1.4 every parallel spin tractor on M naturally gives rise to a series of parallel tractor k−forms, which are nontrivial at least for $k = p+1$. We include all these conformal symmetries in the algebra and use them to construct the odd-odd-bracket. Thus we then also have to modify $\mathfrak{g}_0$ and would like to set

$$\mathfrak{g}_0 := Par\,(\Lambda^*_{\mathcal{T}}(M), \nabla^{nc}) \subset \Omega^*_{\mathcal{T}}(M).$$

One faces two immediate problems:

1. There is no obvious natural generalization of the Lie bracket on $\Lambda^k_{p+1,q+1}$ for $k>2$.

2. The symmetries of $(\psi_1,\psi_2) \mapsto \alpha^k_{\psi_1,\psi_2}$ depend crucially on k.

Our procedure is as in the Lorentzian case:

1. We use the tractor approach and introduce a natural superalgebra structure on the space of parallel forms and twistor spinors.

2. We check Jacobi identities.

3. We describe the algebra wrt. a given metric in the conformal class.

Algebraic preparation

Let us for a moment change our notation to $\mathbb{R}^{r,s}$ and $m = r + s$, as the following results will later be applied to $\mathbb{R}^{p,q}$ and $\mathbb{R}^{p+1,q+1}$. In order to introduce a bracket on $\Lambda^k_{r,s}$, we recall the following formulas for the action of a vector $X \in \mathbb{R}^{r,s}$ and a k-form $\omega \in \Lambda^k_{r,s}$ on a spinor $\varphi \in \Delta^{\mathbb{C}}_{r,s}$ (cf. [BFGK91]):

$$\begin{aligned} X \cdot (\omega \cdot \varphi) &= (X^\flat \wedge \omega) \cdot \varphi - (X \lrcorner \omega) \cdot \varphi, \\ \omega \cdot (X \cdot \varphi) &= (-1)^k \left((X^\flat \wedge \omega) \cdot \varphi + (X \lrcorner \omega) \cdot \varphi \right). \end{aligned} \tag{6.28}$$

This motivates us to set $X \cdot \omega := X^\flat \wedge \omega - X \lrcorner \omega \in \Lambda^{k-1} \oplus \Lambda^{k+1}$ for $X \in \mathbb{R}^{r,s}$ and $\omega \in \Lambda^k_{r,s}$. We use this to set inductively for $e^\flat_I := e^\flat_{i_1} \wedge ... \wedge e^\flat_{i_j} \in \Lambda^j_{r,s}$, where $1 \leq i_1 < i_2 < ... < i_j \leq n$:

$$e^\flat_I \cdot \omega := e_{i_1} \cdot (e^\flat_{I \setminus \{i_1\}} \cdot \omega). \tag{6.29}$$

By multilinear extension, this defines $\eta \cdot \omega \in \Lambda^*_{r,s}$ for all $\eta, \omega \in \Lambda^*_{r,s}$. One checks that this product is associative and $O(r,s)$−equivariant, i.e.

$$(A\eta) \cdot (A\omega) = A(\eta \cdot \omega) \ \forall A \in O(r,s). \tag{6.30}$$

Remark 6.29 The above definition of $\cdot$ is useful for concrete calculations. However, there is an equivalent way of introducing the inner product $\cdot$ on the space of forms which shows that this construction is very natural. To this end, consider the multilinear maps

$$f_k : \underbrace{\mathbb{R}^{r,s} \times ... \times \mathbb{R}^{r,s}}_{k \text{ times}} \to Cl_{r,s}, \ (v_1, ... v_k) \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot v_{\sigma_1} \cdot ... \cdot v_{\sigma_k}.$$

The maps f_k induce a canonical vector space isomorphism (cf. [LM89])

$$\widetilde{f} : \Lambda^*_{r,s} \to Cl_{r,s},$$

for which $\widetilde{f}(v^\flat_1 \wedge ... \wedge v^\flat_k) = f_k(v_1, ..., v_k)$ holds. It is now straightforward to calculate that our inner product (6.29) on $\Lambda^*_{r,s}$ is just the algebra structure which makes $\widetilde{f}$ become an algebra isomorphism, i.e. one has for $\eta, \omega \in \Lambda^*_{r,s}$ that

$$\eta \cdot \omega = \widetilde{f}^{-1} \left(\widetilde{f}(\eta) \cdot \widetilde{f}(\omega) \right). \tag{6.31}$$

With these definitions, the space $\Lambda^*_{r,s}$ together with the map

$$[\cdot, \cdot]_\Lambda : \Lambda^*_{r,s} \otimes \Lambda^*_{r,s} \to \Lambda^*_{r,s}, \ [\eta, \omega]_\Lambda := \eta \cdot \omega - \omega \cdot \eta \tag{6.32}$$

becomes a Lie algebra in a natural way due to associativity of Clifford multiplication.

Remark 6.30 We index this bracket with the subscript Λ because on 2-forms there are now the bracket $[\cdot, \cdot]_\Lambda$ and the endomorphism-bracket $[\cdot, \cdot]_{\mathfrak{so}}$ from the previous sections. However, it is straightforward to calculate that $[\cdot, \cdot]_\Lambda = 2 \cdot [\cdot, \cdot]_{\mathfrak{so}}$. Whence these two Lie algebra structures are equivalent. Note that $[\Lambda^k_{r,s}, \Lambda^l_{r,s}]$ is in general of mixed degree for $k, l \neq 2$.

Conformally invariant definition of $\mathfrak{g}$

Turning to geometry again, let $(M^{p,q},c)$ be a conformal spin manifold of signature (p,q). Given $\alpha,\beta\in\Omega^*_{\mathcal{T}}(M)$ and $x\in M$, we may write $\alpha(x)=[s,\widehat{\alpha}]$ and $\beta(x)=[s,\widehat{\beta}]$ for some $s\in\overline{\mathcal{P}}^1_x$ and $\widehat{\alpha},\widehat{\beta}\in\Lambda^*_{p+1,q+1}$. We then introduce a bracket on tractor forms by setting

$$(\alpha\cdot\beta)(x):=[s,\widehat{\alpha}\cdot\widehat{\beta}]. \tag{6.33}$$

The equivariance property (6.30) shows that (6.33) is well-defined. We furthermore define the bracket $[\alpha,\beta]_{\mathcal{T}}$ on $\Omega^*_{\mathcal{T}}(M)$ by pointwise application of (6.32). Clearly, this defines a Lie algebra structure on $\Omega^*_{\mathcal{T}}(M)$.

Lemma 6.31 *The normal conformal Cartan connection ∇^{nc} on $\Omega^*_{\mathcal{T}}(M)$ is a derivation wrt. the product $\cdot$, i.e.*

$$\nabla^{nc}_X(\alpha\cdot\beta)=(\nabla^{nc}_X\alpha)\cdot\beta+\alpha\cdot(\nabla^{nc}_X\beta)\ \ \forall\alpha,\beta\in\Omega^*_{\mathcal{T}}(M)\ \textit{and}\ X\in\mathfrak{X}(M).$$

Proof. Suppose first that $\alpha=Y^\flat$ for some standard tractor $Y\in\Gamma(\mathcal{T}(M))$. We calculate:

$$\begin{aligned}\nabla^{nc}_X(\alpha\cdot\beta)&=\nabla^{nc}_X(Y^\flat\wedge\beta-Y\lrcorner\beta)\\&=(\nabla^{nc}_XY)^\flat\wedge\beta+Y^\flat\wedge(\nabla^{nc}_X\beta)-(\nabla^{nc}_XY)\lrcorner\beta-Y\lrcorner(\nabla^{nc}_X\beta)\\&=(\nabla^{nc}_X\alpha)\cdot\beta+\alpha\cdot(\nabla^{nc}_X\beta).\end{aligned}$$

As a next step, let $\alpha\in\Omega^*_{\mathcal{T}}(M)$ be arbitrary. We fix $x\in M$ and a local pseudo-orthonormal frame $(s_0,...,s_{n+1})$ (wrt $\langle\cdot,\cdot\rangle_{\mathcal{T}}$) on $\mathcal{T}(M)$ around x such that $\nabla^{nc}s_i=0$ for $i=0,...,n+1$ at x. Wrt. this frame we write $\alpha=\sum_I\alpha_Is_I^\flat$ locally around x for smooth functions α_I. We apply the above result inductively for $Y=s_i$ to obtain at x

$$\begin{aligned}\nabla^{nc}_X(\alpha\cdot\beta)&=\sum_I\nabla^{nc}_X(\alpha_I\cdot s_I\cdot\beta)=\sum_IX(\alpha_I)\cdot s_I\cdot\beta+\sum_I\alpha_Is_I\cdot\nabla^{nc}_X\beta\\&=(\nabla^{nc}_X\alpha)\cdot\beta+\alpha\cdot(\nabla^{nc}_X\beta),\end{aligned}$$

which gives the desired formula. □

Corollary 6.32 *$\alpha,\beta\in Par(\Lambda^*_{\mathcal{T}}(M),\nabla^{nc})$ implies that $[\alpha,\beta]_{\mathcal{T}}\in Par(\Lambda^*_{\mathcal{T}}(M),\nabla^{nc})$. Thus the space $Par(\Lambda^*_{\mathcal{T}}(M),\nabla^{nc})$ together with the bracket induced by $[\cdot,\cdot]_{\mathcal{T}}$ is a Lie subalgebra of $(\Omega_{\mathcal{T}}(M)^*,[\cdot,\cdot]_{\mathcal{T}})$.*

As a next step we extend the Lie algebra of (higher order) conformal symmetries

$$\mathfrak{g}_0:=Par(\Lambda^*_{\mathcal{T}}(M),\nabla^{nc}),$$

together with the bracket defined above to a tractor conformal superalgebra $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ in a natural way by setting as before

$$\mathfrak{g}_1:=Par(\mathcal{S}(M),\nabla^{nc}),$$

and introducing the brackets

$$\begin{aligned}&\mathfrak{g}_0\times\mathfrak{g}_0\to\mathfrak{g}_0,\ (\alpha,\beta)\mapsto[\alpha,\beta]_{\mathcal{T}},\\&\mathfrak{g}_0\times\mathfrak{g}_1\to\mathfrak{g}_1,\ (\alpha,\psi)\mapsto\alpha\cdot\psi,\\&\mathfrak{g}_1\times\mathfrak{g}_0\to\mathfrak{g}_1,\ (\psi,\alpha)\mapsto-\alpha\cdot\psi,\\&\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{g}_0,\ (\psi_1,\psi_2)\mapsto\sum_{l\in L_p}\alpha^l_{\psi_1,\psi_2}.\end{aligned}\tag{6.34}$$

Here, $L_p := \{l \in \mathbb{N} \mid \psi \mapsto \alpha^l_{\psi,\psi}$ not identically 0, $\alpha^l_{\psi_1,\psi_2}$ symm. in $\psi_1, \psi_2\}$, and for given p this set can be completely described with (a complex version of) Remark 1.23. One always has $p+1 \in L_p$, and thus the brackets have the right symmetry properties.

Remark 6.33 If $p = 1$ and we allow only $l = 2$ in the last bracket, we recover a tractor conformal superalgebra which is naturally isomorphic to the one constructed in the previous chapter. Thus, the above construction may be viewed as a reasonable extension to arbitrary signatures.

It is of course natural to ask, as done in the Lorentzian case, under which conditions the tractor conformal superalgebra actually is a *Lie* superalgebra, i.e. we have to check the Jacobi identities:

- As $\mathfrak{g}_0$ is a Lie algebra, the even-even-even identity is trivial.
- It holds by construction of the bracket $[\cdot,\cdot]_{\mathcal{T}}$ as extension of (6.28) that

$$[\alpha,\beta]_{\mathcal{T}} \cdot \psi = \alpha \cdot \beta \cdot \psi - \beta \cdot \alpha \cdot \psi.$$

  But this is precisely the even-even-odd Jacobi identity.
- The Jacobi identity for the odd-odd-odd component again leads to

$$\alpha^l_{\psi,\psi} \cdot \psi \stackrel{!}{=} 0.$$

  However, there is no known way of expressing this condition in terms of $Hol(M,c)$ due to the fact that a classification of possible parallel tractor forms induced by twistor spinors as in Remark 1.25 is only available for the Lorentzian case.
- The even-odd-odd Jacobi identity is by polarization equivalent to

$$[\alpha, \alpha^l_{\psi,\psi}]_{\mathcal{T}} \stackrel{!}{=} 2 \cdot \alpha^l_{\alpha\cdot\psi,\psi} \text{ for } l \in L_p. \tag{6.35}$$

  However, this identity fails to hold in general. From an algebraic point of view this is due to the fact that $[\Lambda^k_{p+1,q+1}, \Lambda^k_{p+1,q+1}]_\Lambda \subset \Lambda^k_{p+1,q+1}$ only if $k = 2$. This was precisely the situation we had in the Lorentzian setting. For other values of k and p the definition of $[\cdot,\cdot]_{\mathcal{T}}$ leads to additional terms on the left hand side of (6.35).

Metric description

As done in the Lorentzian case, we want to compute the brackets of the tractor conformal superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ wrt. a metric in the conformal class. To this end, let $\alpha \in \Lambda^{k+1}_{p+1,q+1}, \beta \in \Lambda^{l+1}_{p+1,q+1}$. As in (2.28) we decompose $\alpha = e^\flat_+ \wedge \alpha_+ + \alpha_0 + e^\flat_- \wedge e^\flat_+ \wedge \alpha_\mp + e^\flat_- \wedge \alpha_-$. We want to compute $([\alpha,\beta]_\Lambda)_+$, i.e. the +-component of $[\alpha,\beta]_\Lambda$ wrt. the decomposition (2.28). As a preparation, we calculate for $\omega \in \Lambda^r_{p,q}, \eta \in \Lambda^s_{p,q}$ the products

$$\begin{aligned}
(e^\flat_\pm \wedge \omega) \cdot \eta &= e^\flat_\pm \wedge (\omega \cdot \eta), \\
(e^\flat_\pm \wedge \omega) \cdot (e^\flat_\pm \wedge \eta) &= 0, \\
(e^\flat_\pm \wedge \omega) \cdot (e^\flat_\mp \wedge \eta) &= (-1)^r \left(e^\flat_\pm \wedge e^\flat_\mp \wedge (\omega \cdot \eta) - \eta \cdot \omega \right), \\
(e^\flat_\pm \wedge \omega) \cdot (e^\flat_- \wedge e^\flat_+ \wedge \eta) &= \mp e^\flat_\pm \wedge (\omega \cdot \eta),
\end{aligned}$$

$$
\begin{aligned}
\omega \cdot (e_\pm^\flat \wedge \eta) &= (-1)^r e_\pm^\flat \wedge (\omega \cdot \eta),\\
\omega \cdot (e_-^\flat \wedge e_+^\flat \wedge \eta) &= e_-^\flat \wedge e_+^\flat \wedge (\omega \cdot \eta),\\
(e_-^\flat \wedge e_+^\flat \wedge \omega) \cdot \eta &= e_-^\flat \wedge e_+^\flat \wedge (\omega \cdot \eta),\\
(e_-^\flat \wedge e_+^\flat \wedge \omega) \cdot (e_\pm^\flat \wedge \eta) &= \pm(-1)^r e_\pm^\flat \wedge (\omega \cdot \eta),\\
(e_-^\flat \wedge e_+^\flat \wedge \omega) \cdot (e_-^\flat \wedge e_+^\flat \wedge \eta) &= \omega \cdot \eta.
\end{aligned}
$$

With these formulas, it is straightforward to compute that for α, β as above one has

$$
(\alpha \cdot \beta)_+ = \alpha_+ \cdot \beta_0 - \alpha_+ \beta_\mp + (-1)^{k+1} \alpha_0 \beta_+ + (-1)^{k+1} \alpha_\mp \cdot \beta_+, \tag{6.36}
$$

and therefore,

$$
\begin{aligned}
([\alpha, \beta]_\Lambda)_+ =&\alpha_+ \cdot \beta_0 - (-1)^{l+1} \beta_0 \cdot \alpha_+ - \alpha_+ \cdot \beta_\mp - (-1)^{l+1} \beta_\mp \cdot \alpha_+ + (-1)^{k+1} \alpha_0 \cdot \beta_+ - \beta_+ \cdot \alpha_0\\
&+ (-1)^{k+1} \alpha_\mp \cdot \beta_+ + \beta_+ \cdot \alpha_\mp.
\end{aligned} \tag{6.37}
$$

This directly leads to the following global version:

Proposition 6.34 *Let $g \in c$ and let $\alpha, \beta \in \mathfrak{g}_0$ be of degree $k+1$ and $l+1$ respectively. Further, let $\alpha_+ = proj^g_{\Lambda,+} \alpha \in \Omega^k_{nc,g}(M)$ and $\beta_+ = proj^g_{\Lambda,+} \beta \in \Omega^l_{nc,g}(M)$ denote the associated nc-Killing forms. As $\alpha \cdot \beta \in \mathfrak{g}_0$ and $[\alpha, \beta]_{\mathcal{T}} \in \mathfrak{g}_0$ are again parallel, the forms $(\alpha \cdot \beta)_+ = proj^g_{\Lambda,+}(\alpha \cdot \beta) \in \Omega^*_{nc,g}(M)$ and $([\alpha, \beta]_{\mathcal{T}})_+ = proj^g_{\Lambda,+}([\alpha, \beta]_{\mathcal{T}}) \in \Omega^*_{nc,g}(M)$ are again nc-Killing forms wrt. g. They are explicitly given by*

$$
\begin{aligned}
\alpha_+ \circ \beta_+ := (\alpha \cdot \beta)_+ =&\frac{1}{l+1} \cdot \alpha_+ \cdot d\beta_+ + \frac{1}{n-l+1} \alpha_+ \cdot d^* \beta_+\\
&+ (-1)^{k+1} \frac{1}{k+1} d\alpha_+ \cdot \beta_+ + (-1)^k \cdot \frac{1}{n-k+1} d^* \alpha_+ \cdot \beta_+,
\end{aligned} \tag{6.38}
$$

$$
\begin{aligned}
([\alpha, \beta]_{\mathcal{T}})_+ =&\frac{1}{l+1} \cdot \alpha_+ \cdot d\beta_+ + (-1)^l \frac{1}{l+1} d\beta_+ \cdot \alpha_+ + \frac{1}{n-l+1} \alpha_+ \cdot d^* \beta_+ + (-1)^{l+1} \frac{1}{n-l+1} d^* \beta_+ \cdot \alpha_+\\
&+ (-1)^{k+1} \frac{1}{k+1} d\alpha_+ \cdot \beta_+ - \frac{1}{k+1} \beta_+ \cdot d\alpha_+ + (-1)^k \cdot \frac{1}{n-k+1} d^* \alpha_+ \cdot \beta_+ - \frac{1}{n-k+1} \beta_+ \cdot d^* \alpha_+.
\end{aligned}
$$

Proof. This follows directly from the explicit form of the isomorphism $proj^g_{\Lambda,+}$ from Theorem 3.7, i.e. one has to insert $(\alpha_+, \alpha_0, \alpha_\mp, \alpha_-) = (\alpha_+, \frac{1}{k+1} d\alpha_+, -\frac{1}{n-k+1} d^* \alpha_+, \Box_k \alpha_+)$ into the formulas (6.36), (6.37). □

We study some interesting consequences and applications. First, note that Proposition (6.34) opens a way to construct new nc-Killing forms out of existing ones, i.e. $\circ$ defines a map

$$
\circ : \Omega^k_{nc,g}(M) \times \Omega^l_{nc,g}(M) \to \Omega^*_{nc,g}(M). \tag{6.39}
$$

In general, the resulting product is of mixed degree. We have already shown in Proposition 6.9 that for nc-Killing 1-forms the bracket $[\cdot, \cdot]_{\mathcal{T}}$ corresponds via fixed $g \in c$ to the Lie bracket of vector fields (up to a factor). For deg $\alpha = 2$ one can simplify the expression from Proposition 6.34 as follows:

Proposition 6.35 *Let $\alpha \in Par(\Lambda^2(M), \nabla^{nc})$, $\beta \in Par(\Lambda^{k+1}(M), \nabla^{nc})$ and $g \in c$. Then it holds for the nc-Killing form $([\beta,\alpha]_{\mathcal{T}})_+ \in \Omega_{nc,g}(M)$ that*

$$\frac{1}{2}([\beta,\alpha]_{\mathcal{T}})_+ = L_{V_\alpha}\beta_+ - (k+1)\lambda_\alpha \cdot \beta_+ \in \Omega^k_{nc,g}(M). \tag{6.40}$$

Here, L denotes the Lie derivative, V_α is the conformal vector field canonically associated to α, and $\lambda_\alpha \in C^\infty(M)$ is defined via $L_{V_\alpha} g = 2\lambda_\alpha \cdot g$. In particular, the right hand side of (6.40) is again a nc-Killing k-form.

Remark 6.36 Proposition 6.35 yields a natural action which gives the space of nc-Killing k-forms the structure of a module for the Lie algebra of normal conformal vector fields. In this context, we remark that it has already been shown in [Sem01] that for a *conformal* vector field V and *conformal* Killing k–form β_+, the form $L_V\beta_+ - (k+1)\lambda_V \cdot \beta_+$ is again a *conformal* Killing k–form.

Proof. Dualizing the first nc-Killing equation (3.4) for α_+ yields

$$\nabla^g_X V_\alpha = (X \lrcorner \alpha_0)^\sharp + \alpha_\mp X. \tag{6.41}$$

We have that $(L_{V_\alpha} g)(X,Y) = g(\nabla^g_X V_\alpha, Y) + g(\nabla^g_Y V_\alpha, X) = 2\lambda_\alpha g(X,Y)$. Inserting (6.41) shows that $\alpha_\mp = \lambda_\alpha \in C^\infty(M)$. We fix $x \in M$ and let $(s_1,...,s_n)$ be a local g–pseudo-orthonormal frame around x. Cartans formula for the Lie derivative L yields that around x we have

$$\begin{aligned}
L_{V_\alpha}\beta_+ &= d\left(V_\alpha \lrcorner \beta\right) + V_\alpha \lrcorner d\beta_+ \\
&= \sum_{i=1}^n \epsilon_i s_i^\flat \wedge \underbrace{\nabla^g_{s_i}\left(V_\alpha \lrcorner \beta_+\right)}_{=(\nabla^g_{s_i} V_\alpha)\lrcorner\beta_+ + V_\alpha \lrcorner \nabla^g_{s_i}\beta_+} + V_\alpha \lrcorner d\beta_+ \\
&= \sum_{i=1}^n \epsilon_i \left(s_i^\flat \wedge \left(\left(\nabla^g_{s_i} V_\alpha\right) \lrcorner \beta_+\right) - V_\alpha \lrcorner \left(s_i^\flat \wedge \nabla^g_{s_i}\beta_+\right) + g(s_i, V_\alpha)\cdot \nabla^g_{s_i}\beta_+\right) + V_\alpha \lrcorner d\beta_+ \\
&= \underbrace{\sum_{i=1}^n \epsilon_i s_i^\flat \wedge \left(\left(\nabla^g_{s_i} V_\alpha\right) \lrcorner \beta_+\right)}_{\mathrm{I}} + \underbrace{\nabla^g_{V_\alpha}\beta_+}_{\mathrm{II}}
\end{aligned}$$

Using the nc-Killing equations for α_+ and β_+, we rewrite the two summands as follows:

$$\mathrm{I} = \underbrace{\sum_{i=1}^n \epsilon_i s_i^\flat \wedge \left((s_i \lrcorner \alpha_0)^\sharp \lrcorner \beta_+\right)}_{\mathrm{I}a} + \underbrace{\alpha_\mp \cdot \sum_{i=1}^n \epsilon_i s_i^\flat \wedge (s_i \lrcorner \beta_+)}_{\mathrm{I}b}$$

Clearly, $\mathrm{I}b = k \cdot \alpha_\mp \cdot \beta_+$. In order to express $\mathrm{I}a$ nicely, we introduce functions a_{ij} such that $(s_i \lrcorner \alpha_0)^\sharp = \sum_j \epsilon_j a_{ij} \cdot s_j$. Clearly, $a_{ij} = -a_{ji}$ and $\alpha_0 = \sum_{i<j} \epsilon_i \epsilon_j a_{ij} s_i^\flat \wedge s_j^\flat$. Inserting this into I$a$ yields that

$$\mathrm{I}a = \sum_{i<j} \epsilon_i \epsilon_j a_{ij} \cdot \left(s_i^\flat \wedge (s_j \lrcorner \beta_+) - s_j^\flat \wedge (s_i \lrcorner \beta_+)\right).$$

In order to simplify this expression, we proceed as follows: Let $s_J^\flat := s_{j_1}^\flat \wedge ... \wedge s_{j_{k+1}}^\flat$ for $1 \leq j_1 < ... < j_{k+1} \leq n$. We compute for $i < j$:

$$\begin{aligned} s_J^\flat \cdot \left(s_i^\flat \wedge s_j^\flat\right) &= \left(s_J^\flat \cdot s_i\right) \cdot s_j = (-1)^{k+1}\left(s_i^\flat \wedge s_J^\flat + s_i \lrcorner s_J^\flat\right) \cdot s_j \\ &= s_i^\flat \wedge s_j^\flat \wedge s_J^\flat + s_i^\flat \wedge \left(s_j \lrcorner s_J^\flat\right) - s_j^\flat \wedge \left(s_i^\flat \lrcorner s_J^\flat\right) + s_i \lrcorner s_j \lrcorner s_J^\flat. \end{aligned}$$

Similarly, one obtains

$$\left(s_i^\flat \wedge s_j^\flat\right) \cdot s_J^\flat = s_i^\flat \wedge s_j^\flat \wedge s_J^\flat - s_i^\flat \wedge \left(s_j^\flat \lrcorner s_J^\flat\right) + s_j^\flat \lrcorner \left(s_i^\flat \wedge s_J^\flat\right) + s_i \lrcorner s_j \lrcorner s_J^\flat.$$

Consequently, $\frac{1}{2} \cdot \left(s_J^\flat \cdot \left(s_i^\flat \wedge s_j^\flat\right) - \left(s_i^\flat \wedge s_j^\flat\right) \cdot s_J^\flat\right) = s_i^\flat \wedge \left(s_j \lrcorner s_J^\flat\right) - s_j^\flat \wedge \left(s_i \lrcorner s_J^\flat\right)$, and multilinear extension immediately yields that

$$\mathrm{I}a = \frac{1}{2}(\beta_+ \cdot \alpha_0 - \alpha_0 \cdot \beta_+).$$

Furthermore, the summand II can with the nc-Killing equation for β_+ be rewritten as

$$\begin{aligned} \nabla^g_{V_\alpha} \beta_+ &= V_\alpha \lrcorner \beta_0 + \alpha \wedge \beta_\mp \\ &= \frac{1}{2} \cdot \left((-1)^{k+1}\beta_0 \cdot \alpha_+ - \alpha_+ \cdot \beta_0\right) + \frac{1}{2} \cdot \left((-1)^{k+1}\beta_\mp \cdot \alpha_+ + \alpha_+ \cdot \beta_\mp\right). \end{aligned}$$

Putting all these formulas together again yields that

$$\begin{aligned} L_{V_\alpha}\beta_+ - (k+1)\lambda_\alpha\beta_+ =& \frac{1}{2}\left((-1)^{k+1}\beta_0 \cdot \alpha_+ - \alpha_+ \cdot \beta_0 + (-1)^{k+1}\beta_\mp \cdot \alpha_+ + \alpha_+ \cdot \beta_\mp + \beta_+ \cdot \alpha_0 - \alpha_0 \cdot \beta_+\right) \\ &+ k \cdot \alpha_\mp \beta_+ - (k+1) \cdot \alpha_\mp \beta_+. \end{aligned}$$

Comparing this expression to (6.37) immediately yields the Proposition. □

As a second application of Proposition 6.34 we consider the case of g being an Einstein metric in the conformal class.

Proposition 6.37 *If $\beta \in \Omega^k_{nc,g}(M)$ is a nc-Killing k-form wrt. an Einstein metric g on M, then both $\beta_0 = (k+1) \cdot d\beta_+$ and $\beta_\mp = -(n-k+1) \cdot d^*\beta_+$ are nc-Killing forms for g as well.*

Proof. As elaborated in [Lei05], on an Einstein manifold (M, g), the tractor 1-form $\alpha = \left(1, 0, 0, -\frac{\mathrm{scal}^g}{2(n-1)n}\right)$ is parallel. Inserting this expression for α into the formulas in Proposition 6.34 shows that $\frac{1}{k+1}d\beta_+ + \frac{1}{n-k+1}d^*\beta_+$ is a nc Killing form. □

Remark 6.38 The last statement has a well-known spinorial analogue: Consider a twistor spinor $\varphi \in \Gamma(S^g)$ on an Einstein manifold. As in this case $\nabla_X D^g\varphi = \frac{n}{2}K^g(X) \cdot \varphi = X \cdot \left(\frac{\mathrm{scal}^g}{4n(n-1)} \cdot \varphi\right)$, the spinor $D^g\varphi$ turns out to be a twistor spinor on (M, g) as well.

We compute the expression of the even-odd bracket wrt. a metric in the conformal class:

Proposition 6.39 *Let $\alpha \in \Omega^{k+1}_T(M)$ be a parallel tractor $(k+1)$-form, $\psi \in \Gamma(\mathcal{S}(M))$ a parallel spin tractor. For given $g \in c$ let $\Phi^g_\Lambda(\alpha) = (\alpha_+, \alpha_0, \alpha_\mp, \alpha_-)$ with $\alpha_+ \in \Omega^k_{nc,g}(M)$*

and $\widetilde{\Phi}^g(\psi) = (\varphi, -\frac{1}{n}D^g\varphi)$ *with* $\varphi \in ker\ P^g$*. Then the twistor spinor corresponding to the parallel spin tractor* $[\alpha,\psi] = \alpha\cdot\psi \in \mathfrak{g}_1$ *via* g *is given by*

$$\begin{aligned}\alpha_+ \circ \varphi := \widetilde{\Phi}^g(proj^g_+(\alpha\cdot\psi)) &= \frac{2}{n}\alpha_+\cdot D^g\varphi + (-1)^{k+1}\alpha_\mp\cdot\varphi + (-1)^{k+1}\alpha_0\cdot\varphi \\ &= \frac{2}{n}\alpha_+\cdot D^g\varphi + \frac{(-1)^k}{n-k+1}d^*\alpha_+\cdot\varphi + \frac{(-1)^{k+1}}{k+1}d\alpha_+\cdot\varphi \in ker\ P^g.\end{aligned}$$

Proof. For given $x\in M$ we consider the reductions $\sigma^g : \mathcal{P}^g_+ \to \overline{\mathcal{P}^1_+}$ and $\widetilde{\sigma}^g : \mathcal{Q}^g_+ \to \overline{\mathcal{Q}^1_+}$ as introduced in chapter 2 with $\sigma^g\circ f^g = \overline{f}^1\circ\widetilde{\sigma}^g$, and on some open neighbourhood U of x in M we have

$$\begin{aligned}\psi &= [\widetilde{\sigma}^g(\widetilde{u}), e_-\cdot w + e_+\cdot w],\\ \alpha &= [\sigma^g(u), \underbrace{e_+^\flat\wedge\widetilde{\alpha}_+ + e_-\wedge e_+\wedge\widetilde{\alpha}_\mp + \widetilde{\alpha}_0 + e_-^\flat\wedge\widetilde{\alpha}_-}_{=:\widetilde{\alpha}}]\end{aligned}$$

for sections $\widetilde{u}: U\to\mathcal{Q}^g_+$, $u = f^g(\widetilde{u}) : U \to \mathcal{P}^g_+$ and smooth functions $w: U\to\Delta_{p+1,q+1}$, $\widetilde{\alpha}_+,\widetilde{\alpha}_- : U\to\Lambda^k_{p,q}$, $\widetilde{\alpha}_0 : U\to\Lambda^{k+1}_{p,q}$ and $\widetilde{\alpha}_\mp : U\to\Lambda^{k-1}_{p,q}$. It follows by definition that on U

$$\alpha\cdot\psi = [\widetilde{\sigma}^g(\widetilde{u}), \widetilde{\alpha}\cdot(e_-\cdot w + e_+\cdot w)].$$

Consequently, we get for the corresponding twistor spinor wrt. g that

$$\widetilde{\Phi}^g(proj^g_+(\alpha\cdot\psi)) = \left[\widetilde{u}, \chi\left(e_-\cdot proj_{Ann(e_+)}(\widetilde{\alpha}\cdot(e_-\cdot w + e_+\cdot w))\right)\right] \tag{6.42}$$

Here, we identify the $Spin(p,q)$–modules $\Delta^{\mathbb{C}}_{p,q}\cong Ann(e_-)$ (cf. (1.10)) by means of some fixed isomorphism χ. One thus has to compute $\widetilde{\alpha}\cdot(e_-\cdot w + e_+\cdot w)$. With the formulas for the action of $\Lambda^*_{p+1,q+1}$ on $\Delta_{p+1,q+1}$, it is straightforward to calculate that this product is given by

$$\begin{aligned}\widetilde{\alpha}\cdot(e_-\cdot w + e_+\cdot w) =& (-1)^k\widetilde{\alpha}_+\cdot e_+\cdot e_-\cdot w + (-1)^k\widetilde{\alpha}_-\cdot e_-\cdot e_+\cdot w + \widetilde{\alpha}_0\cdot(e_-\cdot w + e_+\cdot w)\\ &+ \widetilde{\alpha}_\mp\cdot(e_+\cdot w - e_-\cdot w)\\ =&(e_- + e_+)\cdot((-1)^k e_+\cdot\widetilde{\alpha}_-\cdot w + (-1)^k e_-\cdot\widetilde{\alpha}_+\cdot w + (-1)^{k+1}\widetilde{\alpha}_0\cdot w\\ &+ (-1)^{k+1}\widetilde{\alpha}_\mp\cdot(e_+\cdot e_-\cdot w + w))\\ =:&(e_- + e_+)\cdot\widetilde{w}.\end{aligned}$$

Thus, one has by definition

$$\begin{aligned}\chi\left(e_-\cdot proj_{Ann(e_+)}(\widetilde{\alpha}\cdot(e_-\cdot w + e_+\cdot w))\right) =& \chi(e_-\cdot e_+\cdot\widetilde{w})\\ =& -2\widetilde{\alpha}_+\cdot\chi(e_-\cdot w) + (-1)^{k+1}\cdot\widetilde{\alpha}_0\cdot\chi(e_-\cdot e_+\cdot w)\\ &+ (-1)^{k+1}\cdot\widetilde{\alpha}_\mp\cdot\chi(e_-\cdot e_+\cdot w).\end{aligned}$$

Inserting this into (6.42) yields that

$$\begin{aligned}\widetilde{\Phi}^g(proj^g_+(\alpha\cdot\psi)) =& -2\cdot[u,\widetilde{\alpha}_+]\cdot\underbrace{[\widetilde{u},\chi(e_-\cdot w)]}_{=\widetilde{\Phi}^g(proj^g_-\psi)} + (-1)^{k+1}[u,\widetilde{\alpha}_\mp]\cdot\underbrace{[\widetilde{u},\chi(e_-\cdot e_+\cdot w)]}_{=\widetilde{\Phi}^g(proj^g_+\psi)}\\ &+ (-1)^{k+1}[u,\widetilde{\alpha}_0]\cdot\underbrace{[\widetilde{u},\chi(e_-\cdot e_+\cdot w)]}_{=\widetilde{\Phi}^g(proj^g_+\psi)}\\ =& \frac{2}{n}\alpha_+\cdot D^g\varphi + (-1)^{k+1}\alpha_\mp\cdot\varphi + (-1)^{k+1}\alpha_0\cdot\varphi.\end{aligned}$$

□

Remark 6.40 In particular, Proposition 6.39 describes a principle of constructing new twistor spinors from a given twistor spinor and a nc-Killing form in an arbitrary pseudo-Riemannian setting. One can also show independently and more directly, i.e. without using tractor calculus, that for a given nc-Killing form $\alpha_+ \in \Omega^k_{nc,g}(M)$ and $\varphi \in \ker P^g$, the spinor

$$\alpha_+ \circ \varphi := \frac{2}{n}\alpha_+ \cdot D^g\varphi + \frac{(-1)^k}{n-k+1} d^*\alpha_+ \cdot \varphi + \frac{(-1)^{k+1}}{k+1} d\alpha_+ \cdot \varphi \in \Gamma(S^g) \tag{6.43}$$

is again a twistor spinor on (M,g). To this end, we compute $\nabla^{S^g}_X(\alpha_+ \circ \varphi)$ for $X \in \mathfrak{X}(M)$ using the nc-Killing formulas (3.4):

$$\begin{aligned}
\nabla^{S^g}_X(\alpha_+ \cdot D^g\varphi) &= \left(\nabla^g_X \alpha_+\right) \cdot D^g\varphi + \alpha_+ \cdot \nabla^{S^g}_X D^g\varphi \\
&= (X \lrcorner \alpha_0) \cdot D^g\varphi + (X^\flat \wedge \alpha_\mp) \cdot D^g\varphi + \alpha_+ \cdot \left(\frac{n}{2} \cdot K^g(X) \cdot \varphi\right), \\
\nabla^{S^g}_X(\alpha_0 \cdot \varphi) &= \left(\nabla^g_X \alpha_0\right) \cdot \varphi + \alpha_0 \cdot \nabla^{S^g}_X \varphi \\
&= (K^g(X) \wedge \alpha_+) \cdot \varphi - (X^\flat \wedge \alpha_-) \cdot \varphi - \frac{1}{n} \cdot \alpha_0 \cdot X \cdot D^g\varphi, \\
\nabla^{S^g}_X(\alpha_\mp \cdot \varphi) &= \left(\nabla^g_X \alpha_\mp\right) \cdot \varphi + \alpha_\mp \cdot \nabla^{S^g}_X \varphi \\
&= (K^g(X) \lrcorner \alpha_+) \cdot \varphi + (X \lrcorner \alpha_-) \cdot \varphi - \frac{1}{n} \cdot \alpha_\mp \cdot X \cdot D^g\varphi.
\end{aligned}$$

We deduce using the formulas (6.28) that $\nabla^{S^g}_X(\alpha_+ \circ \varphi) = X \cdot \xi$ for all $X \in \mathfrak{X}(M)$, where $\xi := \left(\frac{1}{n}\alpha_0 \cdot D^g\varphi + \frac{1}{n}\alpha_\mp \cdot D^g\varphi + (-1)^{k+1}\alpha_- \cdot \varphi\right)$, showing that $\alpha_+ \circ \varphi$ satisfies the twistor equation with $D^g(\alpha_+ \circ \varphi) = -n \cdot \xi$.

Finally, we discuss the case of α_+ being a nc-Killing 1-form and V_α the dual normal conformal vector field[5].

Proposition 6.41 *In the setting of Proposition 6.39, if $k = 1$ we have*

$$\Phi^g\left(proj^g_+(\alpha \cdot \psi)\right) = -2 \cdot \underbrace{\left(\nabla_{V_\alpha}\varphi + \frac{1}{4}\tau\left(\nabla V_\alpha\right) \cdot \varphi\right)}_{=: V_\alpha \circ \varphi}, \tag{6.44}$$

where $\tau(\nabla V_\alpha) = \sum_{j=1}^n \epsilon_j \left(\nabla_{s_j} V_\alpha\right) \cdot s_j + (n-2) \cdot \lambda_\alpha$ for any local g–pseudo-orthonormal frame $(s_1, ..., s_n)$, and $L_{V_\alpha} g = 2\lambda_\alpha g$.

Proof. Wrt. g it holds that $\Phi^g(\alpha) = (\alpha_+, \alpha_0, \alpha_\mp, \alpha_-)$. As in the proof of Proposition 6.35 it follows that $\alpha_\mp = \lambda_\alpha$. Let $(s_1, ..., s_n)$ be g–orthonormal. The first nc-Killing equation for α_+ yields that $\nabla^g_{s_j} V_\alpha = (s_j \lrcorner \alpha_0)^\sharp + \alpha_\mp \cdot s_j$. Right-multiplication by s_j gives $\left(\nabla^g_{s_j} V_\alpha\right) \cdot s_j = -(s_j \wedge (s_j \lrcorner \alpha_0)) - \epsilon_j \cdot \alpha_\mp$. Summing over j thus reveals that $\tau(\nabla V_\alpha) = -2 \cdot \alpha_0 - n \cdot \alpha_\mp$, and together with the twistor equation we conclude that the right-hand side of (6.44) is given by

$$-2 \cdot \left(-\frac{1}{n} V_\alpha \cdot D^g\varphi - \frac{1}{2}\alpha_0 \cdot \varphi - \frac{1}{2}\alpha_\mp \cdot \varphi\right).$$

Comparing this to the result of Proposition 6.39 immediately yields (6.44). □

[5] The proof of the following statement is then also the postponed proof of Proposition 6.10.

Remark 6.42 The term $V_\alpha \circ \varphi$ in (6.44) has become standard in the literature as spinorial Lie derivative as introduced in [Kos72, Hab96, Raj06]. Thus, the metric description of the even odd bracket in Proposition 6.39 can be viewed as a generalization of the spinorial Lie derivative to higher order nc-Killing forms, and we see that the brackets in the tractor conformal superalgebra reproduce the spinorial Lie derivative when a metric is fixed. For the case $k = 1$, [Hab96] shows that $X \circ \varphi$ is a twistor spinor for every twistor spinor φ and every *conformal* vector field X, i.e. X need not to be normal conformal.

Remark 6.43 As in the Lorentzian setting, it is also possible in arbitrary signatures to include all conformal Killing forms, i.e. not only nc-Killing forms, in the even part of the algebra in terms of distinguished tractors. However, the generalization of (6.12) to arbitrary signatures, which can be found in [GS08], is technically very demanding.

6.7 Application to special Killing forms on nearly Kähler manifolds

The general relation between special Killing forms and nc-Killing forms

We demonstrate the principle for constructing new nc-Killing forms out of existing ones using the $\circ$–operations from Proposition 6.34 for the case of nearly Kähler manifolds. In this context, we make some more general definitions and remarks:

Definition 6.44 *Let $(M^{p,q}, g)$ be a pseudo-Riemannian manifold of constant scalar curvature $scal^g$. A k–form $\alpha \in \Omega^k(M)$ is called a special Killing k-form to the Killing constant $-\frac{(k+1)scal^g}{n(n-1)}$ if*

$$\begin{aligned} \nabla^g_X \alpha &= \frac{1}{k+1} X \lrcorner d\alpha, \\ \nabla^g_X d\alpha &= -\frac{(k+1)scal^g}{n(n-1)} \cdot X^\flat \wedge \alpha. \end{aligned} \tag{6.45}$$

We let $\Omega^k_{sk,g}(M)$ denote the space of all special Killing k–forms on (M,g).

Examples and classification results for special Killing forms are discussed in [Sem01]. For instance, the dual of every Killing vector field defining a Sasakian structure and the Dirac currents of real Killing spinors on Riemannian manifolds are special Killing 1-forms. Note that every special Killing form is conformal and coclosed, i.e. $d^*\alpha = 0$.

Let us from now on assume that $\mathrm{scal}^g \neq 0$. Under this assumption, spaces carrying special conformal Killing forms can be classified using an analogue of Bärs cone construction for Killing spinors, see [Bär93], for differential forms. More precisely, consider the cone $C(M) = \mathbb{R}^+ \times M$ with cone metric $\widehat{g}_b := bdt^2 + t^2 g$, where $b \neq 0$ is a constant scaling, of signature $(p, q+1)$ or $(p+1, q)$.

Proposition 6.45 *[Sem01] Let $b = \frac{(n-1)n}{scal^g}$. Then special Killing k–forms to the Killing constant $-\frac{(k+1)scal^g}{n(n-1)}$ are in 1-to-1 correspondence to parallel $(k+1)$-forms on the cone*

$(C(M), \widehat{g}_b)$, given by

$$\Omega^k_{sk,g}(M) \ni \alpha \leftrightarrow \widehat{\alpha} := t^k dt \wedge \alpha + \frac{sgn(b)t^{k+1}}{k+1} d\alpha \in \Omega^{k+1}(C(M)) \tag{6.46}$$

Using this, one classifies compact, simply-connected Riemannian manifolds carrying special Killing forms, see [Sem01]. We come back to this list in the last section of this thesis.

Remark 6.46 One can now derive analogous formulas to (6.38),(6.43) for special Killing forms and Killing spinors on pseudo-Riemannian manifolds using the cone construction, i.e. proceed as follows:

1. We let $\alpha \in \Omega^k(M), \beta \in \Omega^l(M)$ be special Killing forms to the same Killing constant and $\varphi \in \Gamma(S^g)$ a Killing spinor on (M,g).
2. Using Bärs construction and Proposition 6.45, we view these objects as parallel tensors $\widehat{\alpha}, \widehat{\beta} \in \Omega^{k+1}(C(M))$, $\widehat{\beta} \in \Omega^{l+1}(C(M))$ and $\widehat{\varphi} \in \Gamma(C(M), S^{\widehat{g}_b})$.
3. We compute $\widehat{\alpha} \cdot \widehat{\beta}$ (with (6.31) applied pointwise) and $\widehat{\alpha} \cdot \widehat{\varphi}$ which again turn out to be parallel forms resp. spinors on the cone.
4. Via (6.46), one expresses these products as special Killing forms resp. Killing spinors on the base (M,g) using the original data α, β, φ and $d\alpha$, $d\beta$ only. Let us call these objects $\alpha \circ \beta \in \Omega^*_{sk,g}(M)$ and $\alpha \circ \varphi \in \mathcal{K}(M)$.

Carrying these steps out is straightforward. One obtains the same formulas (6.38) and (6.43), which of course simplify since $d^*\alpha = 0, D^g\varphi = -\lambda \cdot n \cdot \varphi$ for some $\lambda \in i\mathbb{R} \cup \mathbb{R}$ with $\varphi \in \mathcal{K}_\lambda(M)$. In other words, one obtains a map

$$\begin{aligned} \circ : \Omega^k_{sk,g}(M) \times \Omega^l_{sk,g}(M) &\to \Omega^*_{sk,g}(M), \\ (\alpha, \beta) &\mapsto \alpha \circ \beta = \frac{1}{l+1} \cdot \alpha \cdot d\beta + (-1)^{k+1} \frac{1}{k+1} d\alpha \cdot \beta, \end{aligned} \tag{6.47}$$

and an action of special Killing forms on Killing spinors, given by

$$\begin{aligned} \circ : \Omega^k_{sk,g}(M) \times \mathcal{K}_\lambda(M) &\to \mathcal{K}_\lambda \oplus \mathcal{K}_{-\lambda}(M), \\ (\alpha, \varphi) &\mapsto \alpha \circ \varphi = -2 \cdot \alpha \cdot \varphi + \frac{(-1)^{k+1}}{k+1} d\alpha \cdot \varphi. \end{aligned} \tag{6.48}$$

In particular, (6.47) allows one to *construct new special Killing forms out of existing special Killing forms.*

However, for pseudo-Riemannian Einstein spaces which are not Ricci-flat, special Killing forms are more directly related to normal conformal Killing forms and there is an equivalent way of deriving (6.47) and (6.48):
As seen in (3.15), for every pseudo-Riemannian Einstein space (M,g), the conformal holonomy coincides with the holonomy of an ambient space which is the cone trivially extended by a parallel direction, i.e. $Hol(M,[g]) = Hol(C(M), \widehat{g}_b)$. Using this, it is easy to deduce that there is a natural and bijective correspondence between parallel tractor forms on M, i.e. normal conformal Killing forms for (M,g), and parallel forms on the cone, i.e. special Killing forms for (M,g). More precisely, one shows:

Proposition 6.47 ([Lei07]) *On a pseudo-Riemannian Einstein space of nonvanishing scalar curvature, every nc-Killing form is the sum of a special Killing form and a closed Killing form.*

In particular, the coclosed nc-Killing forms on Einstein spaces are precisely the special Killing forms. This also follows from a direct inspection of the nc-Killing equations (3.4). Note that the well-known spinorial analogue of Proposition 6.47 is the fact that on an Einstein space every twistor spinor decomposes into the sum of two Killing spinors. Thus, for Einstein spaces one obtains the maps (6.47) and (6.48) by restriction of (6.39) and (6.43) to special Killing forms and Killing spinors.

For Einstein spaces, where nc-Killing forms and twistor spinors can be described in terms of special Killing forms and Killing spinors, we can now use the formulas (6.47) and (6.48) to construct new nc-Killing forms out of existing ones which we will do for an example in the next section:

The algebraic structure of special Killing forms on nearly Kähler 6 manifolds

In the following subsection, let (M, g) always be a complete, simply-connected Riemannian manifold.

Definition 6.48 *An almost hermitian manifold (M, g, J) with Kähler form $\omega(X,Y) = g(JX, Y)$ is called* nearly Kähler *manifold if for all $X \in \mathfrak{X}(M)$*

$$X \lrcorner \nabla^g_X \omega = 0.$$

Let us in the following assume that M is nearly Kähler but non-Kähler, i.e. $\nabla J \neq 0$. We say that M is strict nearly Kähler. One can easily see that ω is no special Killing form if the dimension of M is different from 6, whereas every six-dimensional strict nearly Kähler manifold is Einstein with special Killing 2-form ω. For this reason, we specialize to the 6-dimensional case from now on. Let us additionally assume that (M, g) is not isometric to S^6, the standard sphere. Examples for such geometries are studied in [Bär93], for instance. They include $\mathbb{C}P^3$ and $S^3 \times S^3$, equipped with the structure of a Riemannian 3-symmetric space.

Via the correspondence (6.46) the form ω translates to a parallel 3-form $\widehat{\omega}$ on the cone which has stabilizer isomorphic to the exceptional group $G_2 \subset SO(7)$, see [Bär93]. With the above assumptions on M, this shows that the metric cone over (M, g) has holonomy equal to G_2 if and only if the base is strict nearly Kähler. Moreover, the only special Killing forms on (M, g) are given as (cf. [Sem01])

$$\begin{aligned} \Omega^2_{sk,g}(M) \ni \omega &\mapsto \widehat{\omega} \in \Omega^3(C(M)), \\ \Omega^3_{sk,g}(M) \ni *d\omega &\mapsto *\widehat{\omega} \in \Omega^4(C(M)), \end{aligned} \tag{6.49}$$

where the last correspondence is up to a constant. By (6.47), the form $\omega \circ \omega = \frac{1}{3}(\omega \cdot d\omega - d\omega \cdot \omega)$ is again a special-Killing form. By unwinding the definitions, we find the following expression for $\omega \circ \omega$: Let $\omega = \sum_{i,j} \omega_{ij} s_i^\flat \wedge s_j^\flat$ for a local orthonormal basis $(s_1, ..., s_6)$. We have locally that

$$\omega \circ \omega = \frac{2}{3}\sum_{i<j}\omega_{ij}(s_j^\flat \wedge (s_i \lrcorner d\omega) - s_i^\flat \wedge (s_j \lrcorner d\omega)) \in \Omega^3_{sk,g}(M). \tag{6.50}$$

As ω is of Kähler type, (6.50) is never identically zero (note that $d\omega \neq 0$ by the special Killing equations). Using this expression, one can also calculate directly that $\omega\circ\omega$ satisfies the special Killing equations. However, by (6.49) the only special Killing 3-form on M is up to constant given by $*d\omega$. Thus, there exists a nonzero real constant c_1 such that

$$\omega \circ \omega = c_1 \cdot *d\omega. \tag{6.51}$$

Alternatively, one obtains (6.51) by the algebraic computation $\widehat{\omega}\cdot\widehat{\omega} = \text{const.} + c_1 \cdot *\widehat{\omega} \in \Omega^4(C(M))$ and projecting this parallel form to the base again via (6.46). Analogously, one shows that $\omega \circ (*d\omega) \in \Omega^2(M)$, and this product is nonzero[6]. Whence, this special Killing 2-form must again be a constant multiple of ω, i.e there is $c_2 \neq 0$ such that

$$\omega \circ (*d\omega) = c_2 \cdot \omega.$$

Reversing the roles of the two forms shows that

$$(*d\omega) \circ \omega = c_2 \cdot \omega.$$

Here, the fact that $(*d\omega)$ and ω commute under $\circ$ results from the algebraic relation $\widehat{\omega}\cdot(*\widehat{\omega}) = (*\widehat{\omega})\cdot\widehat{\omega}$ on the cone. Finally, an analogous expression to (6.50) reveals that $(*d\omega)\circ(*d\omega) \in \Omega^3(M)$ is nonzero. Whence there is c_3 such that

$$(*d\omega)\circ(*d\omega) = c_3 \cdot (*d\omega).$$

Let us now turn to Killing spinors on M. As shown in [BFGK91], there exists a unique spin structure for (M,g). We work with the associated complex spinor bundle. One proves as in [BFGK91] (directly) or [Bär93] (using the cone correspondence):

Theorem 6.49 *Let (M,g,J) be a simply-connected strict nearly Kähler manifold, $(M,g) \ncong S^6$. Then (M,g) admits real Killing spinors for some Killing number λ. Moreover, the spaces $\mathcal{K}_{\pm\lambda}$ of Killing spinors to Killing number $\pm\lambda$ are one-dimensional and any $\varphi^+ \in \mathcal{K}_{+\lambda}$ satisfies*

$$J(X)\cdot\varphi^+ = iX\cdot\varphi^+ \tag{6.52}$$

for all $X \in \mathfrak{X}(M)$.

Using this, we calculate the action (6.48) of special Killing forms on Killing spinors.

Proposition 6.50 *In the above setting, the map $\circ$ from (6.48) is given as follows: Let $\varphi^+ \in \mathcal{K}_{+\lambda}$ be a nonzero Killing spinor which is unique up to constant. It is easy to see that $\varphi^- := vol\cdot\varphi^+$ is a basis for $\mathcal{K}_{-\lambda}$. We have*

$$\omega\circ\varphi^+ = 2i\lambda\cdot\varphi^+,\ \omega\circ\varphi^- = -2i\lambda\cdot\varphi^-,$$
$$(*d\omega)\circ\varphi^+ = 33i\cdot\lambda^2\cdot\varphi^-,\ (*d\omega)\circ\varphi^- = -33i\cdot\lambda^2\cdot\varphi^+.$$

[6] More precisely, we find that $\omega\circ(*d\omega) \in \Omega^0(M)\oplus\Omega^2(M)\oplus\Omega^6(M)$, where the extremal degree forms are constants and a constant multiple of the volume form, respectively. These are normal conformal Killing forms for trivial reasons and therefore suppressed in the following analysis

Proof. We prove the Proposition for $\varphi := \varphi^+ \in \mathcal{K}_{+\lambda}$. The proof for Killing spinors to the opposite Killing number is analogous. We begin with some later useful identities:
(6.52) implies that

$$\omega \cdot \varphi = -3i\varphi. \tag{6.53}$$

Applying $D^g = \sum_i s_i \cdot \nabla^{S^g}_{s_i}$ to (6.53) and using (6.45) yields

$$d\omega \cdot \varphi = (\sum_i s_i^\flat \wedge \nabla^g_{s_i}\omega) \cdot \varphi = 12i\lambda \cdot \varphi.$$

By (6.48), the action of ω on φ is given by

$$\omega \circ \varphi = -2\lambda \cdot \omega \cdot \varphi - \frac{1}{3} d\omega \cdot \varphi = 6i\lambda \cdot \varphi - 4i\lambda \cdot \varphi = 2i\lambda \cdot \varphi.$$

We turn to the action of $*d\omega$ on φ. To this end, we first remark that a direct computation using only the special Killing equations for ω and $*d\omega$ yields

$$d * d\omega = -\frac{\mathrm{scal}^g}{10} * \omega. \tag{6.54}$$

Furthermore, elementary spinor algebra reveals that for $\eta \in \Lambda^k_6$ a k–form where $k = 2, 3$ and $\chi \in \Delta^{\mathbb{C}}_{0,6}$ a spinor we have

$$(*\eta) \cdot \chi = (-1)^{k+1} \eta \cdot vol_6 \cdot \chi.$$

With these preparations and inserting (6.54) we compute

$$\begin{aligned}
(*d\omega) \circ \varphi &= -2\lambda \cdot (*d\omega) \cdot \varphi - \frac{\mathrm{scal}^g}{40} (*\omega) \cdot \varphi \\
&= -2\lambda \cdot d\omega \cdot vol \cdot \varphi + \frac{\mathrm{scal}^g}{40} \cdot \omega \cdot vol \cdot \varphi \\
&= vol \cdot (2\lambda \cdot d\omega \cdot \varphi - \frac{\mathrm{scal}^g}{40} \cdot \omega \cdot \varphi) \\
&= i \cdot vol \cdot (24\lambda^2 \cdot \varphi + \frac{3 \cdot \mathrm{scal}^g}{40} \cdot \varphi) \\
&= 33i \cdot \lambda^2 \cdot vol \cdot \varphi \in \mathcal{K}_{-\lambda}.
\end{aligned}$$

Here, we used that vol anticommutes with every vector in dimension 6 and the general relation $\mathrm{scal}^g = 4n(n-1) \cdot \lambda^2$ from [BFGK91], where $n = 6$ in our case. □

We summarize our findings:

Theorem 6.51 *Let (M, g, J) be a Riemannian 6-dimensional, complete strict nearly Kähler manifold with $(M, g) \ncong S^6$. Up to linear combinations all special Killing forms on (M, g) are given by ω and $*d\omega$. Moreover, let φ^+ and $\varphi^- = vol \cdot \varphi^+$ denote the up to constant unique Killing spinors on (M, g) to positive resp, negative Killing number. Then the operations $\circ$, as given in (6.47), (6.48), are -up to nonzero constants and by omitting extremal degree forms in the products- given by the following tables:*

$\circ$	ω	$*d\omega$
ω	$*d\omega$	ω
$*d\omega$	ω	$*d\omega$

$\circ$	ω	$*d\omega$
φ^+	φ^+	φ^-
φ^-	φ^-	φ^+

In particular, the special Killing form ω and the Killing spinor φ^+ generate with the actions $\circ$ all other special Killing forms and Killing spinors of (M,g).

For (M,g,J) be a Riemannian 6-dimensional, complete strict nearly Kähler manifold with $(M,g) \ncong S^6$ let us set (cf. also Theorem 6.49)

$$\mathfrak{g}_0 := \Omega_{nc,g}(M) = \text{span}\{\omega, *d\omega\},$$
$$\mathfrak{g}_1 := \mathcal{K}_{+\lambda} \oplus \mathcal{K}_{-\lambda}.$$

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ becomes a superalgebra with (restrictions of) the brackets on $\Omega_{nc,g}(M) \oplus \ker P^g$ as defined before.
By our previous findings, the bracket $[\alpha,\beta]_{\mathcal{T}} = \alpha \circ \beta - \beta \circ \alpha$ on $\Omega_{nc,g}(M)$ is trivial when restricted to special Killing forms $\Omega_{sk,g}(M)$ over (M,g). Moreover, if $\widehat{\omega}$ is the up to constant unique parallel 3-form on $C(M)$ and $\widehat{\varphi}$ the up to constant unique parallel spinor on $C(M)$, an algebraic computation reveals that $\alpha^3_{\widehat{\varphi}} = \widehat{\omega}$ and $\widehat{\omega} \cdot \widehat{\varphi} \neq 0$, unless $\widehat{\varphi} = 0$.
That is, $\mathfrak{g}$ does not satisfy the odd-odd-odd Jacobi identity. Working on the cone, it is also easy to see that the even-odd-odd Jacobi identity is not satisfied: As $\mathfrak{g}_0$ is abelian, it would lead to $\alpha^3_{\widehat{\omega}\cdot\widehat{\varphi},\widehat{\varphi}} = 0$, which is not true.
In particular, the tractor conformal superalgebra associated to a 6-dimensional strict nearly Kähler manifold is no Lie superalgebra.

Remark 6.52 We conclude this example by mentioning some further observations:

1. One can extend the above considerations and include the nc-Killing forms which are not special Killing. By Proposition 6.47 they are given as closed Killing forms.

2. With the same procedure one can calculate analogous tables for weak G_2-manifolds, being 7-dimensional Riemannian manifolds with cone holonomy $Spin(7)$. They turn out to admit distinguished special Killing forms of degree > 1 and Killing spinors as well, see [Bär93, Sem01].

3. The present example shows that the defined actions $\circ$ are in general nontrivial but give a possibility to define a generating system of higher order Killing symmetries (i.e. special Killing forms or Killing spinors).

4. On the space of all differential forms $\Omega(M)$ over a pseudo-Riemannian manifold (M,g), let us *define* the operation $\circ$ by (6.38) for forms of pure degree and extend this to all forms by bilinearity. This operation is associative, and thus defines the Lie bracket $[\alpha,\beta] = \alpha \circ \beta - \beta \circ \alpha$ which *turns $\Omega(M)$ into a Lie algebra.* The preceding discussion then shows that the spaces $\Omega_{nc,g}(M)$ and $\Omega_{sk,g}(M)$ form Lie subalgebras and moreover, on normal conformal 1-forms the bracket reduces to the bracket of vector fields. Thus, we have as a byproduct of the Killing superalgebra construction, found a possible extension of the Lie bracket on vector fields to differential forms on a pseudo-Riemannian manifold.

Remark 6.53 Let us finally mention another tractor conformal superalgebra in non-Lorentzian signature whose properties are governed by a generic 3-form in dimension 7. Consider a conformal spin manifold (M,c) in signature $(4,3)$ admitting a generic real twistor spinor, see section 5.4. Under further generic assumptions on the conformal structure, one has that $Hol(M,c) = G_{2,2} \subset SO^+(4,3)$[7], where $G_{2,2}$ can also be defined as the stabilizer of a generic 3-form $\omega_0 \in \Lambda^3_{4,3}$ under the $SO^+(4,3)$-action, see [Kat99].
Under these conditions, there is up to constant multiples exactly one linearly independent real pure spin tractor $\psi \in \Gamma(M, \mathcal{S}_{\mathbb{R}}(M))$, additionally satisfying dim ker $\psi = 0$. All parallel tractor forms on (M,c) are given by the span of $\alpha^3_\psi \in \Omega^3_{\mathcal{T}}(M)$, being pointwise of type ω_0 and $*\alpha^3_\psi \in \Omega^4_{\mathcal{T}}(M)$, being pointwise of type $*\omega_0$. Thus, the tractor conformal superalgebra of (M,c) is given by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \text{span}\{\alpha^3_\psi, *\alpha^3_\psi\} \oplus \text{span}\{\psi\}.$$

Pure linear algebra in $\mathbb{R}^{4,3}$ reveals that[8]

$$\begin{aligned} \alpha^3_\psi \cdot (*\alpha^3_\psi) &= (*\alpha^3_\psi) \cdot \alpha^3_\psi, \\ \alpha^3_\psi \cdot \psi &= \text{const.}_1 \cdot \psi, \\ (*\alpha^3_\psi) \cdot \psi &= \text{const.}_2 \cdot \psi, \end{aligned}$$

where the ψ–dependent constants are proportional to $\langle \psi, \psi \rangle_{\mathcal{S}}$ and zero iff $\psi = 0$. These observations directly translate into the following properties of the superalgebra $\mathfrak{g}$ with brackets as introduced in (6.34).

Proposition 6.54 *The tractor conformal superalgebra $\mathfrak{g}$ associated to a conformal spin manifold (M,c) in signature $(3,2)$ with $Hol(M,c) = G_{2,2}$ does not satisfy the odd-odd-odd and the even-odd-odd Jacobi identities. Moreover, the even part $\mathfrak{g}_0$ is abelian.*

Again, the same (algebraic) procedure can be carried out for split-signature conformal spin manifolds in dimension 6 with conformal holonomy in $Spin^+(4,3) \subset Spin^+(4,4)$.

The previous examples underline that in contrast to the Lorentzian case, tractor conformal superalgebras need not satisfy at least 3 of the 4 Jacobi identities.

6.8 The possible dimensions of the space of twistor spinors

We have already discussed for Lorentzian signatures, in how far algebraic structures of the tractor conformal superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, in particular, whether it is a *Lie* superalgebra, are related to (local) geometric structures in the conformal class. It is natural to investigate this question further in arbitrary signatures, and we ask ourselves how possible dimensions of the odd *supersymmetric* part $\mathfrak{g}_1$ are related to underlying geometries. Main ingredient is the following algebraic Lemma:

[7] The existence of a generic real twistor spinor always implies $Hol(M,c) \subset G_{2,2}$. The exact conditions for full holonomy $G_{2,2}$ are given in [LN12b] in terms of an explicit ambient metric construction whose metric holonomy coincides with $Hol(M,c)$

[8] See also [Kat99] for explicit formulas of ω_0 and pointwise orbit representatives of ψ.

Lemma 6.55 *For integers r and s consider the bilinear map*

$$V : \Delta^{\mathbb{R}}_{r,s} \otimes \Delta^{\mathbb{R}}_{r,s} \to \mathbb{R}^{r,s},\ (\psi_1, \psi_2) \mapsto V_{\psi_1,\psi_2}$$

mapping a pair of spinors to the associated vector from section 1.4. Let $S_0 \subset \Delta^{\mathbb{R}}_{r,s}$ be a linear subspace and set $V_{S_0} := V_{|S_0 \otimes S_0}$. We have:

1. *If dim $S_0 > \frac{3}{4} \cdot$ dim $\Delta^{\mathbb{R}}_{r,s}$, then V_{S_0} is surjective.*
2. *If dim $S_0 > \frac{1}{2} \cdot$ dim $\Delta^{\mathbb{R}}_{r,s}$, then V_{S_0} is not the zero map.*

Proof. The first part is proved in [AC08]. For the second part, assume that $V_{S_0}(\psi_1, \psi_2) = 0$ for all $\psi_1, \psi_2 \in S_0$. By definition, this is equivalent to $\langle v \cdot \psi_1, \psi_2 \rangle_{\Delta^{\mathbb{R}}_{r,s}} = 0$ for all $\psi_1, \psi_2 \in S_0$ and $v \in \mathbb{R}^{r,s}$, i.e. $\forall v \in \mathbb{R}^{r,s} : cl(v) : S_0 \to S_0^{\perp}$. As dim $S_0 > \frac{1}{2} \cdot$ dim $\Delta^{\mathbb{R}}_{r,s}$, it follows that dim $S_0^{\perp} < \frac{1}{2} \cdot$ dim $\Delta^{\mathbb{R}}_{r,s}$. Thus the map $cl(v)$ has a kernel for every $v \in \mathbb{R}^{r,s}$, i.e. there is $\psi_v \in \Delta^{\mathbb{R}}_{r,s} \backslash \{0\}$ with $v \cdot \psi_v = 0$. This implies that $\langle v, v \rangle_{r,s} = 0$ for every $v \in \mathbb{R}^{r,s}$. □

Remark 6.56 The second statement in Lemma 6.55 cannot be improved in general. Namely, taking $r = s = 2$ and $S_0 := \Delta^{\mathbb{R},\pm}_{2,2} \subset \Delta^{\mathbb{R}}_{2,2}$ provides an example for dim $S_0 = \frac{1}{2} \cdot$dim $\Delta^{\mathbb{R}}_{r,s}$ and $V_{S_0} = 0$.

Applications of Lemma 6.55 have already been studied in the literature::

Proposition 6.57 *[AC08] Let $(M^{p,q}, g)$ be a pseudo-Riemannian spin manifold of dimension n with real spinor bundle $S^g = S^g_{\mathbb{R}}(M)$ of rank N.*

1. *If (M, g) admits $k > \frac{3}{4} N$ twistor spinors which are linearly independent at $x \in M$, then (M, g) admits n conformal vector fields, which are linearly independent at $x \in M$.*
2. *If (M, g) admits $k > \frac{3}{4} N$ parallel spinors, then (M, g) is flat.*

We now apply Lemma 6.55 in the tractor setting yielding a conformal analogue of the second part of Proposition 6.57. Let $(M^{p,q}, c)$ be a conformal spin structure with real spin tractor bundle $\mathcal{S}(M)$ and space of *real* twistor spinors $\mathfrak{g}_1$. Let $N_c := 2 \cdot$dim $\Delta^{\mathbb{R}}_{p,q}$ denote the rank of $\mathcal{S}(M)$, which is the maximal number of linearly independent real twistor spinors on (M, c).

Proposition 6.58 *In the above notation, we have:*

1. *If dim $\mathfrak{g}_1 > \frac{3}{4} \cdot N_c$, then (M, c) is conformally flat.*
2. *If dim $\mathfrak{g}_1 > \frac{1}{2} \cdot N_c$, then there exists an Einstein metric in c (at least on an open and dense subset).*

Proof. We apply Lemma 6.55 to the case that $r = p + 1$, $s = q + 1$ and $S_0 \subset \Delta^{\mathbb{R}}_{p+1,q+1}$ being the subspace of $Hol(M, c)$-invariant spinors (for some fixed base points) which as we know correspond to twistor spinors. Surjectivity of V yields a basis of $\mathbb{R}^{p+1,q+1}$ which is $Hol(M, c)$-invariant. This proves the first part.

For the second part, it follows analogously by nontriviality of V_{S_0} that there exists at least one nontrivial holonomy-invariant vector. By Theorem 3.16 this yields an Einstein scale in the conformal class (on an open, dense subset). □

Remark 6.59 As a simply-connected, conformally flat manifold always admits the maximal number of twistor spinors, the previous Proposition implies that either dim $\mathfrak{g}_1 \leq \frac{3}{4} \cdot N_c$ or dim $g_1 = N_c$ is maximal, i.e. the dimension of $\mathfrak{g}_1$ cannot be arbitrary for the simply-connected case.

In the second case of Proposition 6.58 one can say more: To this end, let $(M^n, c = [g])$ be a simply-connected pseudo-Riemannian conformal spin manifold where g is a Ricci-flat metric. Let k denote the number of linearly independent parallel vector fields on (M,g). As a direct consequence of (3.16) we have for $x \in M$ that

$$\mathfrak{hol}_x(M,[g]) = \mathfrak{hol}_x(M,g) \ltimes \mathbb{R}^{n-k} = \left\{ \begin{pmatrix} 0 & v^\flat & 0 \\ 0 & A & -v \\ 0 & 0 & 0 \end{pmatrix} \mid A \in \mathfrak{hol}_x(M,g), v \in \mathbb{R}^{n-k} \right\},$$

where the matrix is written wrt. the basis $(s_+, s_1, ..., s_n, s_-)$ of $\mathcal{T}_x(M)$ for some pseudo-orthonormal basis $(s_1,...,s_n)$ of T_xM. Assume now that $k < n$, i.e. (M,g) is Ricci-flat but non flat, and let $\psi \in \mathfrak{g}_1$ be a parallel spin tractor on $(M,[g])$ with twistor spinor φ. It follows by the holonomy-principle that

$$\lambda_*^{-1}\left(\begin{pmatrix} 0 & v^\flat & 0 \\ 0 & 0 & -v \\ 0 & 0 & 0 \end{pmatrix} \right) \cdot \psi(x) = 0, \tag{6.55}$$

for all $v \in \mathbb{R}^{n-k} \subset \mathbb{R}^n$, i.e. $s_+ \cdot v \cdot \psi(x) = 0$ (cf. [Fis13] for formulas for λ_*^{-1} in this situation). Choosing v to be non lightlike yields that $s_+ \cdot \psi(x) = 0$ for all $x \in M$ which is equivalent to $D^g\varphi = -n \cdot \widetilde{\Phi}^g(proj^g_-\psi) = 0$. Thus, φ is a parallel spinor on (M,g), ker $\psi \neq \{0\}$, and we have proved:

Proposition 6.60 *Let (M,g) be a simply-connected Ricci-flat spin manifold. Then either every twistor spinor on (M,g) is parallel or (M,g) is flat. In particular, if for a conformal structure (M,c) there is a Ricci-flat metric in the conformal class and dim $\mathfrak{g}_1$ is not maximal, then $\mathfrak{g}$ is a Lie superalgebra.*

We now come back to the second case of Proposition 6.58: It follows now directly from Proposition 6.60 that in case of dim $\mathfrak{g}_1 > \frac{1}{2} \cdot \left(2 \cdot \text{dim}\Delta^{\mathbb{R}}_{p,q}\right)$ there exists an Einstein metric in c with nonzero scalar curvature or the conformal structure is conformally flat, provided that M is simply-connected.

Example 6.61 We consider a special class of conformally Ricci-flat Lorentzian metrics admitting twistor spinors, namely plane waves (M,h) which are equivalently characterized by the existence of local coordinates $(x, y_1, ..., y_n, z)$ such that

$$h = 2dxdz + \left(\sum_{i,j=1}^n a_{ij} y_i y_j \right) dz^2 + \sum_{i=1}^n dy_i^2,$$

where the a_{ij} are functions only of z. It is $Ric^h = \sum_{i=1}^n a_{ii}dz^2$ and the isotropic vector field $\frac{\partial}{\partial x}$ is parallel. Let us assume that (M,h) is indecomposable. Then it is known from [Lei06] that for x in M

$$\mathfrak{hol}_x(M,[h]) = \mathbb{R}^{2n+1} = \left\{ \begin{pmatrix} 0 & 0 & u^T & c & 0 \\ 0 & 0 & v^T & 0 & -c \\ 0 & 0 & 0 & -v & -u \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid u, v \in \mathbb{R}^n, c \in \mathbb{R} \right\}$$

This explicit description makes it straightforward to calculate all spinors annihilated by $\lambda_*^{-1}(\mathfrak{hol}_x(M,[h]))$, yielding that dim ker $P^g = \frac{1}{2} \cdot$ dim $\Delta^{\mathbb{R}}_{1,n+1} = \frac{1}{4} \cdot N_c$.

7 Charged Conformal Killing Spinors

In the last part of this thesis, we will elaborate on how the natural generalization of already studied $Spin^c$-spinor field equations and the twistor equation together with the question of what the spinorial analogue of conformal, not necessarily normal conformal vector fields might be, leads to the study of the twistor equation on pseudo-Riemannian $Spin^c$-manifolds. That is why this final chapter is devoted to the classification and construction of geometries admitting solutions to the equation

$$\nabla^A_X \varphi + \frac{1}{n} X \cdot D^A \varphi = 0 \text{ for all } X \in \mathfrak{X}(M), \tag{7.1}$$

whose objects will be defined shortly. Let us call solutions of (7.1) $Spin^c$–twistor spinors or charged conformal Killing spinors (CCKS).

As we shall see, applying the theory of Cartan connections and tractor calculus does not lead to a significant simplification of the classification problem of determining conformal geometries admitting $Spin^c$-twistor spinors. Therefore, our methods in this chapter are quite elementary and its main parts are self-contained.
We first introduce the basic ingredients of conformal $Spin^c$–geometry in arbitrary signature and show how CCKS can be described as parallel sections in the double spinor bundle wrt. a suitable connection. We then investigate the integrability conditions resulting from the CCKS equation, the relations between the Weyl curvature and the curvature of the S^1–connection and the properties of the spinor bilinears constructed out of a CCKS. Clearly, all these results can be viewed as generalizations of formulas for the $Spin$–setting which have appeared before. Section 7.4 is then devoted to CCKS on Fefferman spaces which is precisely the $Spin^c$–analogue of [Bau99]. Based on the results obtained so far, we can then present a partial classification result in section 7.5. In section 7.6 we continue the local analysis of the CCKS equation which has been initiated recently in physics literature and end up with a local geometric description of geometries admitting CCKS in signatures $(1,4)$, $(0,5), (2,2)$ and $(3,2)$. Finally, we use our results to comment on the relation between conformal and normal conformal vector fields in section 7.7.

7.1 $Spin^c$-geometry and the twistor operator

$Spin^c(p,q)$-groups and spinor representations

This is a brief review of algebraic aspects of $Spin^c$–geometry. The real and complex Clifford algebras and spinor modules have already been introduced in section 1.2.

Remark 7.1 In this chapter, we work with the following concrete realisation of an irreducible, complex representation of $Cl^{\mathbb{C}}_{p,q}$: Let E, T, U and V denote the 2×2 matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Furthermore, let $\tau_j = \begin{cases} 1 & \epsilon_j = 1, \\ i & \epsilon_j = -1. \end{cases}$. Let $n = 2m$. In this case, $Cl^{\mathbb{C}}(p,q) \cong M_{2^m}(\mathbb{C})$ as complex algebras, and an explicit realisation of this isomorphism is given by

$$\begin{aligned} \Phi_{p,q}(e_{2j-1}) &= \tau_{2j-1} E \otimes ... \otimes E \otimes U \otimes T \otimes ... \otimes T, \\ \Phi_{p,q}(e_{2j}) &= \tau_{2j} E \otimes ... \otimes E \otimes V \otimes \underbrace{T \otimes ... \otimes T}_{(j-1)\times}. \end{aligned}$$

Let $n = 2m+1$ and $q > 0$. In this case, there is an isomorphism $\widetilde{\Phi}_{p,q} : Cl^{\mathbb{C}}(p,q) \to M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$, given by

$$\begin{aligned} \widetilde{\Phi}_{p,q}(e_j) &= (\Phi_{p,q-1}(e_j), \Phi_{p,q-1}(e_j)),\ j = 1, ..., 2m, \\ \widetilde{\Phi}_{p,q}(e_{2m+1}) &= \tau_{2m+1}(iT \otimes ... \otimes T, -iT \otimes ... \otimes T), \end{aligned}$$

and $\Phi_{p,q} := pr_1 \circ \widetilde{\Phi}_{p,q}$ is an irreducible representation mapping $\omega_{\mathbb{C}}$ to Id.

The Clifford group contains $Spin^+(p,q)$ as well as the unit circle $S^1 \subset \mathbb{C}$ as subgroups. Together they generate the group $Spin^c(p,q)$ and since $S^1 \cap Spin^+(p,q) = \{\pm 1\}$, we have

$$Spin^c(p,q) = Spin^+(p,q) \cdot S^1 = Spin^+(p,q) \times_{\mathbb{Z}_2} S^1.$$

$Spin^c(p,q)$ has various algebraic relations to other groups, see [Fri00]. We let $\lambda : Spin^+(p,q) \to SO^+(p,q)$ denote the 2-fold covering. There are natural maps

$$\begin{aligned} \lambda^c &: Spin^c(p,q) \to SO^+(p,q),\ [g,z] \mapsto \lambda(g), \\ \zeta &: Spin^c(p,q) \to SO^+(p,q) \times S^1,\ [g,z] \mapsto (\lambda(g), z^2), \end{aligned} \tag{7.2}$$

where ζ is a 2-fold covering. The Lie algebra of $Spin^c(p,q)$ is thus given by $\mathfrak{spin}^c(p,q) = \mathfrak{spin}(p,q) \oplus i\mathbb{R}$. ζ_* turns out to be a Lie algebra isomorphism, given by $\zeta_*(e_i \cdot e_j, it) = (2E_{ij}, 2it)$, where $E_{ij} = -\epsilon_j D_{ij} + \epsilon_i D_{ji}$ for the standard basis D_{ij} of $\mathfrak{gl}(n,\mathbb{R})$.
For $(p,q) = (2p', 2q')$, the group $Spin^c(p,q)$ is related to the group $U(p',q')$ of pseudo-unitary matrices as follows: Let $\iota : \mathfrak{gl}(m,\mathbb{C}) \ni A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{gl}(2m,\mathbb{R})$ denote the natural inclusion and define $F : U(p',q') \to SO(p,q) \times S^1$ by $F(A) = (\iota A, \det A)$. Then there is exactly one group homomorphism $l : U(p',q') \to Spin^c(p,q)$ such that $\zeta \circ l = F$, i.e. the following diagram commutes:

$$\begin{tikzcd} Spin^c(p,q) \arrow[dr, "\zeta"] & \\ U(p',q') \arrow[u, "l"] \arrow[r, "F"] & SO(p,q) \times S^1 \end{tikzcd}$$

For $n = 2m$ or $n = 2m+1$, fixing an irreducible complex representation $\rho : Cl^{\mathbb{C}}_{p,q} \to End\left(\Delta^{\mathbb{C}}_{p,q}\right)$ on the space of spinors $\Delta^{\mathbb{C}}_{p,q} = \mathbb{C}^{2^m}$, for instance $\rho = \Phi$ from Remark 7.1, and restricting it to $Spin^c(p,q) \subset Cl^{\mathbb{C}}_{p,q}$ yields the complex spinor representation

$$\rho : Spin^c(p,q) \to End\left(\Delta^{\mathbb{C}}_{p,q}\right),\ \rho([g,z])(v) = z \cdot \rho(g)(v) =: z \cdot g \cdot v.$$

In case n odd, the restrictions of the two irreducible Clifford representations to $Spin^c(p,q)$ coincide and yield an irreducible representation whereas in case $n=2m$ even $\Delta^{\mathbb{C}}_{p,q}$ splits into the sum of two inequivalent $Spin^c(p,q)$-representations $\Delta^{\mathbb{C},\pm}_{p,q}$. In our realisation from Remark 7.1, let us denote by $u(\delta)\in\mathbb{C}^2$ the vector $u(\delta)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-\delta i\end{pmatrix}$, $\delta=\pm1$, and set $u(\delta_1,...,\delta_m):=u(\delta_1)\otimes...\otimes u(\delta_m)\in\Delta^{\mathbb{C}}_{p,q}$ for $\delta_j=\pm1$. Then $\Delta^{\mathbb{C},\pm}_{p,q}=\text{span}\{u(\delta_1,...,\delta_m)\mid\prod_{j=1}^m\delta_j=\pm1\}$ for n even.
In this chapter we work with the Hermitian inner product $\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{C}}_{p,q}}$ on the spinor module $\Delta^{\mathbb{C}}_{p,q}=\mathbb{C}^{2^m}$ given by

$$\langle u,v\rangle_{\Delta^{\mathbb{C}}_{p,q}}=d\cdot(e_1\cdot...\cdot e_p\cdot u,v)_{\mathbb{C}^{2^m}},$$

where $(u,v):=\sum_j^{2^m}u_j\overline{v}_j$, d is some power of i depending on p,q and the concrete realisation of the representation only. In the realisation from Remark 7.1 we take $d=i^{p(p-1)/2}$. Lemma 1.13 applies also for this situation and in particular, $\langle\cdot,\cdot\rangle_{\Delta^{\mathbb{C}}_{p,q}}$ is invariant under $Spin^c(p,q)$.

Moreover, as in the $Spin$–setting, bilinears can be constructed out of spinors. Concretely, we associate to spinors $\chi_{1,2}\in\Delta^{\mathbb{C}}_{p,q}$ a series of forms $\alpha^k_{\chi_1,\chi_2}\in\Lambda^k_{p,q}$, $k\in\mathbb{N}$, given by

$$\langle\alpha^k_{\chi_1,\chi_2},\alpha\rangle_{p,q}:=d_{k,p}\left(\langle\alpha\cdot\chi_1,\chi_2\rangle_{\Delta_{p,q}}\right)\ \forall\alpha\in\Lambda^k_{p,q}. \tag{7.3}$$

The map $d_{k,p}:\mathbb{C}\to\mathbb{C}$ has been introduced in section 1.4. We set $\alpha^k_\chi:=\alpha^k_{\chi,\chi}$. In more invariant notation these forms arise in even dimension as the image of a pair of spinors under the map

$$\Delta^{\mathbb{C}}_{p,q}\otimes\Delta^{\mathbb{C}}_{p,q}\overset{\langle\cdot,\cdot\rangle}{\to}End(\Delta^{\mathbb{C}}_{p,q})\cong Cl^{\mathbb{C}}(p,q)\cong\left(\Lambda^*_{p,q}\right)^{\mathbb{C}}\to\Lambda^k_{p,q}.$$

Remark 7.2 For $\chi\in\Delta^{\mathbb{C}}_{p,q}$ and $k\in\mathbb{N}$ we have that $\alpha^p_\chi=0\Leftrightarrow\chi=0$. Moreover, α^k_χ is explicitly given by $\alpha^k_\chi=\sum_{1\leq i_1<i_2<...<i_k\leq n}\epsilon_{i_1}...\epsilon_{i_k}d_{k,p}\left(\langle e_{i_1}\cdot...e_{i_k}\cdot\chi,\chi\rangle_{\Delta^{\mathbb{C}}_{p,q}}\right)e^\flat_{i_1}\wedge...\wedge e^\flat_{i_k}$ and the equivariance property

$$\alpha^k_{z\cdot g\cdot\chi}=\lambda(g)(\alpha^k_\chi)$$

holds for all $k\in\mathbb{N}$, $z\cdot g\in Spin^c(p,q)$.

$Spin^c$-structures and spinor bundles

Let (M,g) be a space-and time-oriented, connected pseudo-Riemannian manifold of index p and dimension $n=p+q\geq3$. As usual, by $\mathcal{P}^g_+$ we denote the $SO^+(p,q)$-principal bundle of all space-and time-oriented pseudo-orthonormal frames.

Definition 7.3 *A $Spin^c$-structure of (M,g) is given by the data $(\mathcal{Q}^c,\mathcal{P}_1,f^c)$, where $\mathcal{P}_1$ is a S^1-principal bundle over M, $\mathcal{Q}^c$ is a $Spin^c(p,q)$-principal bundle over M which together with $f^c:\mathcal{Q}^c\to\mathcal{P}^g_+\times\mathcal{P}_1$ defines a ζ–reduction of the product $SO^+(p,q)\times S^1$-bundle $\mathcal{P}^g_+\times\mathcal{P}_1$ to $Spin^c(p,q)$.*

Existence and uniqueness of $Spin^c$–structures is discussed elsewhere, see [LM89]. We will from now on assume that (M,g) admits a $Spin^c$–structure (which is locally always guaranteed) and assume that this structure is fixed. In order to stress the metric, we also write

$\mathcal{Q}^c_g$ instead of $\mathcal{Q}^c$. Given a $Spin^c$-manifold, the associated bundle $S^g = \mathcal{Q}^c \times_{Spin^c(p,q)} \Delta^{\mathbb{C}}_{p,q}$ is called the complex spinor bundle. In case n even, it holds that $S^g = S^{g,+} \oplus S^{g,-}$.

The algebraic objects introduced in the last section define fibrewise Clifford multiplication $\mu : \Omega^*(M) \otimes S^g \to S^g$ and an Hermitian inner product $\langle \cdot, \cdot \rangle_{S^g}$. Moreover, pointwise applying the construction of spinor bilinears (7.3) leads to series of differential forms $\Gamma(M, S^g) \otimes \Gamma(M, S^g) \to \Omega^k(M)$ associated to pairs of spinor fields. Dualizing this for $k = 1$, leads to the well-known Dirac current $V_\varphi \in \mathfrak{X}(M)$. This is in complete analogy to the construction for the *Spin*-case which appeared before several times.

Let $\omega^g \in \Omega^1(\mathcal{P}^g_+, \mathfrak{so}(p,q))$ denote the Levi-Civita connection ∇^g on (M,g), considered as a bundle connection. In addition, we fix a connection $A \in \Omega^1(\mathcal{P}_1, i\mathbb{R})$ in the S^1-bundle. Together, they form a connection $\omega^g \times A$ on $\mathcal{P}^g_+ \times \mathcal{P}_1$, which lifts to

$$\widetilde{\omega^g \times A} := \zeta_*^{-1} \circ (\omega^g \times A) \circ df^c \in \Omega^1(\mathcal{Q}^c, \mathfrak{spin}^c(p,q)).$$

The covariant derivative ∇^A on S^g induced by this connection is locally given as follows: One writes $\varphi \in \Gamma(S^g)$ locally as $\varphi_{|U} = [\widetilde{s \times e}, v]$, where $s \in \Gamma(U, \mathcal{P}^g_+), e \in \Gamma(U, \mathcal{P}_1)$ and $\widetilde{s \times e}$ is a lifting to $\Gamma(U, \mathcal{Q}^c)$. We have on U:

$$\nabla^A_X \varphi_{|U} = [\widetilde{s \times e}, X(v) + \frac{1}{2} \sum_{1 \leq k < l \leq n} \underbrace{\epsilon_k \epsilon_l g(\nabla^g_X s_k, s_l)}_{=:\omega_{kl}(X)} e_k \cdot e_l \cdot v + \frac{1}{2} A^e(X) \cdot v] \tag{7.4}$$

The inclusion of a S^1-connection A in the construction of this covariant derivative "*gauges*" the natural S^1-action on S^g, by which we mean the following: Let $f = e^{i\tau/2} : M \to S^1$ be a smooth function. Then (7.4) directly implies that

$$\nabla^A_X (f \cdot \varphi) = \frac{i}{2} d\tau(X) \cdot f \cdot \varphi + f \cdot \nabla^A_X \varphi = f \cdot \nabla^{A + id\tau}_X \varphi. \tag{7.5}$$

It is moreover known from [Fri00] that for all $X, Y \in \mathfrak{X}(M)$ and $\varphi, \psi \in \Gamma(S^g)$ we have

$$\begin{aligned} \nabla^A_X (Y \cdot \varphi) &= \nabla^g_X Y \cdot \varphi + Y \cdot \nabla^A_X \varphi, \\ X \langle \varphi, \psi \rangle_{S^g} &= \langle \nabla^A_X \varphi, \psi \rangle_{S^g} + \langle \varphi, \nabla^A_X \psi \rangle_{S^g}. \end{aligned}$$

The curvature form $F_A = dA$ of A can be seen as element of $\Omega^2(M, i\mathbb{R})$. Let R^A denote the curvature tensor of ∇^A and $R^g : \Lambda^2(TM) \to \Lambda^2(TM)$ the curvature tensor of (M,g). It holds that (see [Fri00])

$$\begin{aligned} R^A(X,Y)\varphi &= \frac{1}{2} R^g(X,Y) \cdot \varphi + \frac{1}{2} dA(X,Y) \cdot \varphi, \\ \sum_i \epsilon_i s_i \cdot R^A(s_i, X)\varphi &= \frac{1}{2} Ric(X) \cdot \varphi - \frac{1}{2} (X \lrcorner dA) \cdot \varphi. \end{aligned} \tag{7.6}$$

Remark 7.4 $Spin^c$-structures arise naturally in the following situations:

1. Every spin manifold is canonically $Spin^c$ with trivial S^1–bundle. Taking for A the canonical flat connection on $M \times S^1$ in this situation makes ∇^A correspond to the *Spin*–connection on S^g induced by the Levi-Civita connection, see [Mor97].

2. Let M be a manifold which admits a $U(p',q') \hookrightarrow SO^+(p,q)$ reduction $(\mathcal{P}_U, h : \mathcal{P}_U \to \mathcal{P}^g_+)$ of its frame bundle. Then the bundles $(\mathcal{Q}^c := \mathcal{P}_U \times_l Spin^c(p,q), \mathcal{P}_1 := \mathcal{P}_U \times_{\det} S^1)$ together with the map

$$f^c : \mathcal{Q}^c \to \mathcal{P}^g_+ \times \mathcal{P}_1,\ [q, z \cdot g]_l \mapsto ([q, \lambda(g)], [q, z^2])$$

define a $Spin^c(p,q)$-structure on M. In this situation, there are natural reduction maps

$$\begin{aligned} \phi_c &: \mathcal{P}_U \to \mathcal{Q}^c,\ p \mapsto [p,1]_l, \\ \phi_1 &: \mathcal{P}_U \to \mathcal{P}_1,\ p \mapsto [p,1]_{\det}. \end{aligned}$$

Moreover, local sections in $\mathcal{Q}^c$ can be obtained as follows: Let $s \in \Gamma(V, \mathcal{P}_U)$ be a local section on $V \subset M$. Then we have that $\phi_c(s) \in \Gamma(V, \mathcal{Q}^c)$ and

$$f^c(\phi_c(s)) = s \times e, \text{ where } e = \phi_1(s). \tag{7.7}$$

Basic properties of charged conformal Killing spinors

Given a pseudo-Riemannian $Spin^c$-manifold (M,g) together with a connection A on the underlying S^1-bundle, there are naturally associated differential operators. The composition of ∇^A with Clifford multiplication defines the Dirac operator

$$D^A : \Gamma(S^g) \overset{\nabla^A}{\to} \Gamma(T^*M \otimes S^g) \overset{g}{\cong} \Gamma(TM \otimes S^g) \overset{\mu}{\to} \Gamma(S^g),$$

for which the Schröder-Lichnerowicz formula (cf. [Fri00]) yields the relation

$$D^{A,2}\varphi = \Delta^A \varphi + \frac{R}{4}\varphi + \frac{1}{2} dA \cdot \varphi, \tag{7.8}$$

where $\Delta^A \varphi = -\sum_i \epsilon_i \left(\nabla^A_{e_i} \nabla^A_{e_i} \varphi - \operatorname{div}(e_i) \nabla^A_{e_i} \varphi \right)$ and $R = \mathrm{scal}^g$ is the scalar curvature of (M,g). A complementary operator is obtained by performing the spinor covariant derivative ∇^A followed by orthogonal projection onto the kernel of Clifford multiplication. This gives rise to the $Spin^c$-twistor operator P^A

$$P^A : \Gamma(S^g) \overset{\nabla^A}{\to} \Gamma(T^*M \otimes S^g) \overset{g}{\cong} \Gamma(TM \otimes S^g) \overset{proj_{\ker \mu}}{\to} \Gamma(\ker \mu).$$

Spinor fields $\varphi \in \ker P^A$ are called $Spin^c$-twistor spinors. A local calculation reveals that they are equivalently characterized as solutions of the $Spin^c$-twistor equation

$$\nabla^A_X \varphi + \frac{1}{n} X \cdot D^A \varphi = 0 \text{ for all } X \in \mathfrak{X}(M).$$

Following the conventions in [CKM+14, HTZ13, KTZ12], we shall call $Spin^c$-twistor spinors charged conformal Killing spinors and abbreviate them by CCKS.

In analogy to the $Spin$-case, CCKS are objects of conformal $Spin^c$–geometry: Let $f^c_g : \mathcal{Q}^c_g \to \mathcal{P}^g_+ \times \mathcal{P}_1$ be a $Spin^c(p,q)$-structure for (M,g) and let $\widetilde{g} = e^{2\sigma} g$ be a conformally equivalent metric. As in the case of spin structures (cf. section 2.3), there exists a canonically induced $Spin^c$–structure $f^c_{\widetilde{g}} : \mathcal{Q}^c_{\widetilde{g}} \to \mathcal{P}^{\widetilde{g}}_+ \times \mathcal{P}_1$ and a $Spin^c(p,q)$-equivariant map $\widetilde{\phi}_\sigma : \mathcal{Q}^c_g \to \mathcal{Q}^c_{\widetilde{g}}$ such that the diagram

$$\begin{tikzcd} \mathcal{Q}^c_g \arrow[r, "\widetilde{\phi}_\sigma"] \arrow[d, "f^c_g"'] & \mathcal{Q}^c_{\widetilde{g}} \arrow[d, "f^c_{\widetilde{g}}"] \\ \mathcal{P}^g_+ \times \mathcal{P}_1 \arrow[r, "\phi_\sigma"'] & \mathcal{P}^{\widetilde{g}}_+ \times \mathcal{P}_1 \end{tikzcd}$$

commutes, where $\phi_\sigma((s_1,...,s_n),e) = ((e^{-\sigma}s_1,...,e^{-\sigma}s_n),e)$. We obtain identifications

$$\begin{array}{llllcll} \widetilde{\ }: S^g & \to & S^{\widetilde{g}}, & \varphi & = & [\widehat{q},v] & \mapsto & [\widetilde{\phi}_\sigma(\widehat{q}),v] = \widetilde{\varphi}, \\ \widetilde{\ }: TM & \to & TM, & X & = & [q,x] & \mapsto & [\phi_\sigma(q),x] = e^{-\sigma}X, \end{array}$$

where the second map is an isometry wrt. g and $\widetilde{g}$. With these identifications, the covariant derivative ∇^A on the spinor bundle, the Dirac operator and the twistor operator transform in the following way (the proof is the same as in the $Spin$-case, see [BFGK91, Bau81]):

$$\begin{aligned} \nabla^{A,\widetilde{g}}_{\widetilde{X}}\widetilde{\varphi} &= e^{-\sigma}\widetilde{\nabla^{A,g}_X\varphi} - \frac{1}{2}e^{-2\sigma}(X\cdot\text{grad}^g(e^\sigma)\cdot\varphi + g(X,\text{grad}^g(e^\sigma))\cdot\varphi)\widetilde{\ } \\ D^{A,\widetilde{g}}\widetilde{\varphi} &= e^{-\frac{n+1}{2}\sigma}\left(D^{A,g}(e^{\frac{n-1}{2}\sigma}\varphi)\right)\widetilde{\ } \\ P^{A,\widetilde{g}}\widetilde{\varphi} &= e^{-\frac{\sigma}{2}}\left(P^{A,g}(e^{-\frac{\sigma}{2}}\varphi)\right)\widetilde{\ } \end{aligned}$$

Thus, $P^{A,g}$ is conformally covariant and $\varphi \in \ker P^{A,g}$ iff $e^{\sigma/2}\widetilde{\varphi} \in \ker P^{A,\widetilde{g}}$. The S^1-bundle data, and in particular A, are unaffected by the conformal change. However, applying (7.5) directly yields the following additional S^1–gauge invariance of the CCKS-equation:

Proposition 7.5 *Let $\varphi \in ker\ P^{A,g}$ and $f = e^{i\tau/2} \in C^\infty(M,S^1)$. Then $f\varphi \in ker\ P^{A-id\tau,g}$ and $D^{A-id\tau}(f\varphi) = fD^A\varphi$.*

Consequently, the data needed to define CCKSs are in fact a conformal manifold $(M,[g])$, where we require that (M,g) is $Spin^c$ for one - and hence for all - $g \in c$, and a gauge equivalence class of S^1-connections in the underlying S^1-bundle $\mathcal{P}_1$.

Proposition 7.6 *The following hold for $\varphi \in ker\ P^{A,g}$:*

$$D^{A,2}\varphi = \frac{n}{n-1}\left(\frac{R}{4}\varphi + \frac{1}{2}dA\cdot\varphi\right), \tag{7.9}$$

$$\nabla^A_X D^A\varphi = \frac{n}{2}\left(K^g(X) + \frac{1}{n-2}\cdot\left(\frac{1}{n-1}X\cdot dA + X\lrcorner dA\right)\right)\cdot\varphi. \tag{7.10}$$

Proof. All calculations are carried out at a fixed point $x \in M$. Let $(s_1,...,s_n)$ be a pseudo-orthonormal frame which is parallel in x. We have at x:

$$-\Delta^A\varphi + \frac{1}{n}D^{A,2}\varphi = \sum_i \epsilon_i\nabla^A_{s_i}\left(\nabla^A_{s_i}\varphi + \frac{1}{n}s_i\cdot D^A\varphi\right) = 0,$$

and thus by (7.8) $\frac{1}{n}D^{A,2}\varphi = \Delta^A\varphi = D^{A,2}\varphi - \frac{R}{4}\varphi - \frac{1}{2}dA\cdot\varphi$, from which (7.9) follows. To prove (7.10), note that the twistor equation yields $R^A(X,s_i)\varphi = -\frac{1}{n}\left(s_i\nabla^A_X D^A\varphi - X\cdot\nabla^A_{s_i}D^A\varphi\right)$, for X a vector field which is parallel in x. Inserting this into (7.6) implies that

$$\begin{aligned} Ric(X)\cdot\varphi &= \frac{2}{n}(2-n)\nabla^A_X D^A\varphi + \frac{2}{n}X\cdot D^{A,2}\varphi + (X\lrcorner dA)\cdot\varphi \\ &= \frac{2}{n}(2-n)\nabla^A_X D^A\varphi + \frac{R}{2(n-1)}X\cdot\varphi + \frac{1}{n-1}X\cdot dA\cdot\varphi + (X\lrcorner dA)\cdot\varphi. \end{aligned}$$

Solving for $\nabla^A_X D^A\varphi$ yields the claim. □

Proposition 7.6 leads to an equivalent characterization of CCKS. To this end, consider the bundle $E^g := S^g \oplus S^g$ together with the covariant derivative

$$\nabla^{E^g,A}_X \begin{pmatrix}\varphi\\ \psi\end{pmatrix} := \begin{pmatrix} \nabla^A_X\varphi + \frac{1}{n}X\cdot\psi \\ \nabla^A_X\psi - \frac{n}{2}\left(K^g(X) + \frac{1}{n-2}\cdot\left(\frac{1}{n-1}X\cdot dA + X\lrcorner dA\right)\right)\cdot\varphi \end{pmatrix}.$$

Obviously, $\varphi\in\ker P^A$ implies that $\nabla^{E^g,A}\begin{pmatrix}\varphi\\ D^A\varphi\end{pmatrix} = 0$, and on the other hand, if $\nabla^{E^g,A}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = 0$, then $\varphi\in\ker P^A$ and $\psi = D^A\varphi$. It follows as in the *Spin*-case that for a nontrivial CCKS the spinors φ and $D^A\varphi$ never vanish at the same point and $\dim\ker P^A \le 2^{[n/2]+1}$.

7.2 CCKS and conformal Cartan geometry

Given a pseudo-Riemannian $Spin^c$-manifold (M,g) with auxiliary S^1–bundle $\mathcal{P}_1$, we saw in the previous section that a conformal change of the metric induces a natural identification of the associated $Spin^c$-bundles. The S^1–data are unaffected by the conformal change. Let us rephrase this in terms of principal bundles: We define the conformal $Spin^c$-group of signature (p,q) to be

$$CSpin^c(p,q) := \mathbb{R}^+ \times Spin^c(p,q) \cong CSpin^+(p,q)\times_{\mathbb{Z}_2} S^1.$$

Clearly, the map λ^0 as given in (2.12) extends to a natural double covering

$$\begin{aligned} \zeta_0 : CSpin^c(p,q) &\to CO^+(p,q)\times S^1, \\ [g,z] &\mapsto (\lambda^0(g), z^2). \end{aligned}$$

Definition 7.7 *Let (M,c) be a space-and time-oriented conformal manifold of signature (p,q) with $CO^+(p,q)$-bundle $\mathcal{P}^0_+$ of conformal orthonormal frames. A conformal $Spin^c$-structure of signature (p,q) over (M,c) is given by a S^1-principal bundle $\mathcal{P}_1$ and a ζ_0-reduction $(\mathcal{Q}^c_0, f^c_0)$ of the $CO^+(p,q)\times S^1$-bundle $\mathcal{P}^0_+\times\mathcal{P}_1$. (M,c) together with a conformal $Spin^c$-structure is called a conformal $Spin^c$-manifold.*

Clearly, a conformal $Spin^c$–structure induces a $Spin^c$–structure with the same S^1–bundle for every metric $g\in c$ by setting $\mathcal{Q}^c_g := (f^c_0)^{-1}(\mathcal{P}^g_+\times\mathcal{P}_1)$ and $f^c_g := f^c_{0|\mathcal{Q}^c_g}$. On the other hand, given a $Spin^c$-structure $(\mathcal{Q}^c_g, f^c_g)$ for (M,g), we obtain a conformal $Spin^c$–structure for $(M,[g])$ as follows: We observe that $\mathcal{P}^0_+\times\mathcal{P}_1 \cong (\mathcal{P}^g_+\times\mathcal{P}_1)\times_{(SO^+(p,q)\times S^1)}(CO^+(p,q)\times S^1)$ and then get the conformal $Spin^c$-structure $(\mathcal{Q}^c_g\times_{Spin^c(p,q)} CSpin^c(p,q), f^c_g\times\zeta_0)$ by trivial enlargement of the $Spin^c$-structure for (M,g).

In this section, we want to describe conformal $Spin^c$-structures in terms of Cartan geometries and investigate the CCKS-equation in this setting.
First, one has to understand the flat model for conformal $Spin^c$-structures. Recall from (2.21) the groups $B^+ = SO^+(p+1,q+1)$ and $P^+ = Stab_{\mathbb{R}^+e_-}B^+ = B_0^+ \ltimes B_1^+$ as well as their preimages $\widetilde{B}^+$ and $\widetilde{P}^+$ under the double covering map λ. The space $\widehat{Q}^{p,q}$, which is the space of all null-rays in $\mathbb{R}^{p+1,q+1}$ is isomorphic to $\widetilde{B}^+/\widetilde{P}^+$ and the natural Cartan geometry $(\widetilde{B}^+ \to \widetilde{B}^+/\widetilde{P}^+, \widetilde{\omega}^{MC})$ can be viewed as $\widehat{Q}^{p,q}$ equipped with its natural conformally flat class of metrics and a natural conformal spin structure as explained in chapter 2.

Let us now consider the extended double covering (7.2) in signature $(p+1,q+1)$

$$\zeta : \widetilde{B}^{+,c} := Spin^c(p+1,q+1) \to SO^+(p+1,q+1) \times S^1 \tag{7.11}$$

and introduce the group $\widetilde{P}^{+,c} := \zeta^{-1}\left(P^+ \times S^1\right)$. Clearly, $\widehat{Q}^{p,q} \cong \widetilde{B}^{+,c}/\widetilde{P}^{+,c}$ and we view the resulting Cartan geometry

$$(\widetilde{B}^{+,c} \to \widehat{Q}^{p,q}, \widetilde{\omega}^{MC}_{\widetilde{B}^{+,c}})$$

of type $(\widetilde{B}^{+,c}, \widetilde{P}^{+,c})$ as the flat model for conformal $Spin^c$–structures. This is reasonable because obviously, the Maurer Cartan form $\widetilde{\omega}^{MC}_{\widetilde{B}^{+,c}} \in \Omega^1(Spin^c(p+1,q+1), \mathfrak{spin}^c(p+1,q+1))$ of this Cartan geometry is the lift of $\omega^{MC}_{B^+\times S^1} \in \Omega^1(SO^+(p+1,q+1)\times S^1, \mathfrak{so}(p+1,q+1)\oplus i\mathbb{R})$ to $Spin^c(p+1,q+1)$ via (7.11), where $\omega^{MC}_{B^+\times S^1}$ splits into the Maurer Cartan form of $SO^+(p+1,q+1)$, which is identified with the normal conformal Cartan connection on $\widehat{Q}^{p,q}$, and $\omega^{MC}_{S^1} \in \Omega^1(S^1, i\mathbb{R})$ which is identified with the canonical flat S^1–connection on $\mathcal{P}_1 = \widehat{Q}^{p,q} \times S^1$.

Remark 7.8 On the level of Lie algebras, the flat model arises from an extension of the grading of $\mathfrak{b} = \mathfrak{so}(p+1,q+1)$ from (2.13) by inclusion of a central part $i\mathbb{R}$,

$$\mathfrak{b} \oplus i\mathbb{R} = \mathfrak{b}_{-1} \oplus (\mathfrak{b}_0 \oplus i\mathbb{R}) \oplus \mathfrak{b}_1.$$

Clearly, $LA(\widetilde{P}^{+,c}) = (\mathfrak{b}_0 \oplus i\mathbb{R}) \oplus \mathfrak{b}_1$. Thus, the first prolongation for arbitrary conformal $Spin^c$-structures has to start with the $CSpin^c(p,q)$-bundle $\mathcal{Q}_0^c$ and produce a Cartan geometry of type $(\widetilde{B}^{+,c}, \widetilde{P}^{+,c})$.

Returning to the general setting, we consider a conformal $Spin^c$-manifold (M,c) of signature (p,q) with S^1-bundle $\mathcal{P}_1$ and double covering $f_0^c : \mathcal{Q}_0^c \to \mathcal{P}_+^0 \times \mathcal{P}_1$. Let $pr_{1,2}$ denote the projection of f_0^c to the corresponding factor of $\mathcal{P}_+^0 \times \mathcal{P}_1$. We carry out the first prolongation for the underlying conformal structure $\mathcal{P}_+^0$ as discussed in section 2.4, yielding the P^+-principal bundle $\mathcal{P}_+^1 = \{H \subset T_u\mathcal{P}_+^0 \mid u \in \mathcal{P}_+^0, H \text{ horizontal and torsion free}\}$ and then introduce

$$\mathcal{Q}_c^1 := \{H \subset T_q\mathcal{Q}_0^c \mid q \in \mathcal{Q}_0^c, H \text{ horizontal and } d(pr_1 \circ f_0^c)_q(H) \in \mathcal{P}_+^1\}.$$

We define a $\widetilde{P}^{+,c}$-right action on this space as follows: For $\widetilde{b} = \widetilde{b}_0 \cdot \widetilde{b}_1 \in \widetilde{P}^{+,c} \cong CSpin^c(p,q) \ltimes \widetilde{B}_1^+ \subset \widetilde{B}^{+,c}$ and $H \in \mathcal{Q}_c^1$, $H \subset T_q\mathcal{Q}_0^c$ we set

$$H \cdot \widetilde{b} := (df^c_{0\,q\cdot\widetilde{b}_0})^{-1}(d(pr_1 \circ f_0^c)_q(H) \cdot \lambda^c(\widetilde{b})).$$

Clearly, this generalizes the definition for the $Spin$–case from section 2.7 and checking the following facts is straightforward:

Proposition 7.9 *The above defined action together with the projection* $\pi_{\mathcal{Q}^1_c} : \mathcal{Q}^1_c \ni H_q \mapsto \pi_{\mathcal{Q}^c_0}(q) \in M$ *turns* $\mathcal{Q}^1_c$ *into a* $\widetilde{P}^{+,c}$*-principal bundle over* M*. Moreover, the map*

$$\begin{aligned} f^1_c : \mathcal{Q}^1_c &\to \mathcal{P}^1_+ \times \mathcal{P}_1, \\ H_q &\mapsto (d(pr_1 \circ f^c_0)_q(H), pr_2(f^c_0(q))) \end{aligned}$$

is a double covering and $(\mathcal{Q}^1_c, f^1_c)$ *is a* $\lambda^c_{|\widetilde{P}^{+,c}} : \widetilde{P}^{+,c} \to P^+ \times S^1$*-reduction of* $\mathcal{P}^1_+ \times \mathcal{P}_1$.

Let $\omega^{nc} \in \Omega^1(\mathcal{P}^1_+, \mathfrak{so}(p+1,q+1))$ denote the normal conformal Cartan connection on $\mathcal{P}^1_+$. Together with a fixed S^1–connection A on the bundle $\mathcal{P}_1$, which belongs to the data of a conformal $Spin^c$–structure we started with, we obtain a Cartan connection $\widetilde{\omega^{nc} \times A}$ on the $\widetilde{P}^{+,c}$-bundle $\mathcal{Q}^1_c$ by demanding commutativity of the following diagram:

$$\begin{tikzcd} T\mathcal{Q}^1_c \arrow[r, "\widetilde{\omega^{nc}\times A}"] \arrow[d, "df^1_c"'] & \mathfrak{spin}^c(p+1,q+1) \arrow[d, "\zeta_*"] \\ T(\mathcal{P}^1_+ \times \mathcal{P}_1) \arrow[r, "\omega^{nc}\times A"] & \mathfrak{so}(p+1,q+1) \oplus i\mathbb{R} \end{tikzcd}$$

Consequently, we have constructed a Cartan geometry $(\mathcal{Q}^1_c \to M, \widetilde{\omega^{nc} \times A})$ of type $(\widetilde{B}^{+,c}, \widetilde{P}^{+,c})$ out of a conformal $Spin^c$–structure $(\mathcal{Q}^c_0, \mathcal{P}_1, f^c_0)$ with fixed connection A on the underlying S^1–bundle $\mathcal{P}_1$.

In analogy to the $Spin$–case, we define a $Spin^c$-tractor bundle by

$$\mathcal{S}^c(M) := \mathcal{Q}^1_c \times_{\widetilde{P}^{+,c}} \Delta^{\mathbb{C}}_{p+1,q+1}.$$

Let us turn to metric descriptions of these objects: Fixing a metric $g \in c$ leads to a $SO^+(p,q) \times S^1$–reduction $\sigma^g_c = \sigma^g \times Id$ of $\mathcal{P}^1_+ \times \mathcal{P}_1$, i.e.

$$\sigma^g_c : \mathcal{P}^g_+ \times \mathcal{P}_1 \to \mathcal{P}^1_+ \times \mathcal{P}_1,\ (u,v) \mapsto (\ker\, \omega^g_u, v).$$

Again, ω^g also denotes the extension of the Levi-Civita connection to a torsion-free connection on $\mathcal{P}^0_+$. If moreover a connection A on $\mathcal{P}_1$ is fixed, one obtains the $Spin^c(p,q) \hookrightarrow Spin^c(p+1,q+1)$-reduction

$$\widetilde{\sigma}^g_{c,A} : \mathcal{Q}^c_g \to \mathcal{Q}^1_c,\ q \mapsto \ker\left(\widetilde{\omega^g \times A}\right)_q.$$

The construction makes the following diagram become commutative:

$$\begin{tikzcd} \mathcal{Q}^c_g \arrow[r, "\widetilde{\sigma}^g_{c,A}"] \arrow[d, "f^c_g"'] & \mathcal{Q}^1_c \arrow[d, "f^1_c"] \\ \mathcal{P}^g_+ \times \mathcal{P}_1 \arrow[r, "\sigma^g_c"'] & \mathcal{P}^1_+ \times \mathcal{P}_1 \end{tikzcd} \tag{7.12}$$

In complete analogy to (2.33) every metric $g \in c$ leads to an isomorphism

$$\begin{aligned} \widetilde{\Phi}^g_{c,A} : \mathcal{S}^c(M) &\to S^g(M) \oplus S^g(M), \\ [\widetilde{\sigma}^g_{c,A}(l), e_- w + e_+ w] &\mapsto [l, \zeta(e_+ w)] + [l, \chi(e_- w)], \end{aligned}$$

with the algebraic maps χ and ζ from (1.11). The canonical extension of $\widetilde{\omega^{nc}\times A}$ to a principal bundle connection on the extended $Spin^c(p+1,q+1)$−principal bundle $\overline{\mathcal{Q}}^1_c$ induces a covariant derivative $\nabla^{nc,A}$ on the associated vector bundle $\mathcal{S}^c(M)\cong\overline{\mathcal{Q}}^1_c\times_{Spin^c(p+1,q+1)}\Delta^{\mathbb{C}}_{p+1,q+1}$ in the usual way.

Proposition 7.10 *For $g\in c$, the $g-$metric representation of $\nabla^{nc,A}$, i.e. the map $\widetilde{\Phi}^g_{c,A}\circ\nabla^{nc,A}\circ\left(\widetilde{\Phi}^g_{c,A}\right)^{-1}$ is given by*

$$\nabla^{nc,A}_X\begin{pmatrix}\varphi\\ \phi\end{pmatrix}=\begin{pmatrix}\nabla^{S^g,A}_X & -X\cdot\\ \frac{1}{2}K^g(X)\cdot & \nabla^{S^g,A}_X\end{pmatrix}\begin{pmatrix}\varphi\\ \phi\end{pmatrix}.$$

Proof. In this proof, we identify $\mathcal{Q}^1_c$ with a subset of $\overline{\mathcal{Q}}^1_c$ and denote ω^{nc} as well as its canonical extension by the same symbol.
Let $s\times e:U\to\mathcal{P}^g_+\times\mathcal{P}_1$ be a local section with lift $\widetilde{s\times e}:U\to\mathcal{Q}^g_c$. For any $\psi\in\Gamma(U,\mathcal{S}^c(M))$ we find a smooth function $v:U\to\Delta^{\mathbb{C}}_{p+1,q+1}$ such that $\psi=[\widetilde{\sigma}^g_{c,A}(\widetilde{s\times e}),v]$. Further, let $\rho:Spin^c(p+1,q+1)\to\Delta^{\mathbb{C}}_{p+1,q+1}$ denote the standard representation. We have on U:

$$\begin{aligned}\nabla^{nc,A}_X\psi&=[\widetilde{\sigma}^g_{c,A}(\widetilde{s\times e}),X(v)+\rho_*\underbrace{((\widetilde{\omega^{nc}\times A})(d(\widetilde{\sigma}^g_{c,A}(\widetilde{s\times e}))(X)))}_{\overset{(7.12)}{=}\zeta_*^{-1}((\omega^{nc}\times A)(d(\sigma^g(s)\times e)(X)))}v]\\&=[\widetilde{\sigma}^g_{c,A}(\widetilde{s\times e}),X(v)+\rho_*(\lambda_*^{-1}(\omega^{nc}(d(\sigma^g(s))(X))))(v)+\frac{1}{2}A^e(X)\cdot v]\end{aligned}$$

For the middle term in the above expression we insert (2.20) for the metric expression of ω^{nc} to obtain together with the local formula (7.4) a $Spin^c$-analogue of (2.35), i.e.

$$\nabla^{nc,A}_X=\nabla^{S^g,A}_X+\widetilde{\rho}^g(X)+\widetilde{\rho}^g(K^g(X)),\tag{7.13}$$

where $\widetilde{\rho}$ is given by (2.34) with the obvious modifications to the $Spin^c$-setting. Evaluating the last two summands in (7.13) is purely algebraic and has already been carried out in the proof of Proposition 2.34. With this, we arrive at the desired formula. □

Let us apply this to $Spin^c$−twistor spinors. (7.10) shows that CCKS wrt. $g\in c$ are equivalently characterized as sections $(\varphi,\phi)\in\Gamma(S^g\oplus S^g)$ satisfying

$$\begin{pmatrix}\nabla^{S^g,A}_X & -X\cdot\\ \frac{1}{2}K^g(X)\cdot & \nabla^{S^g,A}_X\end{pmatrix}\begin{pmatrix}\varphi\\ \phi\end{pmatrix}=\begin{pmatrix}0\\ E_{dA}(X)\cdot\varphi\end{pmatrix},$$

where the dA−dependent endomorphism $E_{dA}:TM\to\Lambda^*TM\otimes_{\mathbb{R}}\mathbb{C}$ is given by $E_{dA}(X)=-\frac{1}{2(n-1)}\left(\frac{1}{n-2}X^\flat\wedge dA+X\lrcorner dA\right)$. In this case, $\phi=-\frac{1}{n}D^A\varphi$.
For fixed $g\in c$, the tractor s_- which lies in the canonical isotropic line $\mathcal{I}_-$ of $\mathcal{T}(M)\overset{g}{\cong}\mathcal{I}_-\oplus TM\oplus\mathcal{I}_+$, see (2.23), acts on $S^g\oplus S^g$ as $s_-\cdot\begin{pmatrix}\varphi\\ \phi\end{pmatrix}=\begin{pmatrix}0\\ \varphi\end{pmatrix}$ as follows from (1.11). We now define wrt. some $g\in c$

$$\begin{aligned}&F_{dA}\in\Omega^1(M,\Lambda^*_{\mathcal{T}}(M)\otimes_{\mathbb{R}}\mathbb{C}),\\&F_{dA}(X)\overset{g\in c}{=}E_{dA}(X)\wedge s_-^\flat.\end{aligned}$$

As a consequence of the transformation formula (2.27) this is well-defined, i.e. $E_{dA}(X)\wedge s_-^\flat$ transforms as a tractor form under a conformal change. With Proposition 7.10 and these observations, we immediately obtain a characterization of CCKS in terms of conformally invariant objects:

Theorem 7.11 *Let (M,c) be a conformal $Spin^c$-manifold. For $g \in c$ the spinor $\varphi \in \Gamma(S^g)$ is a CCKS wrt. the S^1-connection A iff the tractor $\psi := \left(\widetilde{\Phi}^g_{c,A}\right)^{-1}\begin{pmatrix}\varphi \\ -\frac{1}{n}D^A\varphi\end{pmatrix} \in \Gamma(\mathcal{S}^c(M))$ satisfies*

$$\nabla^{nc,A}_X\psi = F_{dA}(X)\cdot\psi. \tag{7.14}$$

Remark 7.12 The previous observations reveal that in terms of tractors CCKS are *not* characterized as parallel objects but as *generalized Killing spinors on the $Spin^c-$tractor bundle*. Equivalently, they are parallel tractors wrt. the *modified* connection $\nabla^{nc,A} - F_{dA}$. It is easy to verify that $F_{dA} = 0$ iff $dA = 0$.
It does not seem that (7.14) leads to a reasonable simplification of the study of the $Spin^c$ twistor equation as it has no direct consequences for the conformal holonomy group $Hol(M,c)$. Therefore, we will not use tractor calculus in the following investigation of charged conformal Killing spinors.

7.3 Integrability conditions and spinor bilinears

CCKS are parallel wrt. the *ad-hoc* defined connection $\nabla^{E^g,A}$. We obtain integrability conditions for the existence of CCKS by computing the curvature operator $R^{\nabla^{E^g,A}}$ which has to vanish when applied to $(\varphi, D^A\varphi)^T$, where $\varphi \in \ker P^{A,g}$. Let $pr_{1,2}$ denote the projections onto the corresponding summands of E^g. We calculate:

$$\begin{aligned} pr_1\left(R^{\nabla^{E^g,A}}(X,Y)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right) =& \frac{1}{2}\left(R^g(X,Y) - X\cdot K^g(Y) + Y\cdot K^g(X)\right)\cdot\varphi + \frac{1}{2}dA(X,Y)\cdot\varphi \\ & - \frac{1}{2(n-2)}\left(\frac{1}{n-1}(X\cdot Y - Y\cdot X)\cdot dA + (X\cdot(Y\lrcorner dA) - Y\cdot(X\lrcorner dA))\right)\cdot\varphi \end{aligned}$$

With the definition of the Weyl tensor W^g and using the identities

$$\begin{aligned} X\cdot\omega &= X^\flat\wedge\omega - X\lrcorner\omega, \\ \omega\cdot X &= (-1)^k\left(X^\flat\wedge\omega + X\lrcorner\omega\right), \end{aligned} \tag{7.15}$$

where X is a vector and ω a $k-$form, we obtain the integrability condition

$$\begin{aligned} 0 =& \frac{1}{2}\cdot W^g(X,Y)\cdot\varphi + \left(\frac{n-3}{2(n-1)}\cdot dA(X,Y) - \frac{1}{(n-2)(n-1)}\cdot X^\flat\wedge Y^\flat\wedge dA\right)\cdot\varphi \\ & + \frac{1}{n-2}\left(\frac{1}{n-1} - \frac{1}{2}\right)\cdot\left(X^\flat\wedge(Y\lrcorner dA) - Y^\flat\wedge(X\lrcorner dA)\right)\cdot\varphi. \end{aligned} \tag{7.16}$$

In particular, $\ker P^A$ being of maximal possible dimension implies $W^g = 0$ and $dA = 0$. The integrability condition resulting from $pr_2\left(R^{\nabla^{E^g,A}}(X,Y)\begin{pmatrix}\varphi\\ D^A\varphi\end{pmatrix}\right) = 0$ is with the same formulas and $C^g(X,Y) = (\nabla^g_X K^g)(Y) - (\nabla^g_Y K^g)(X)$, straightforwardly calculated to be

$$
\begin{aligned}
0 =& \frac{1}{2}W^g(X,Y)\cdot D^A\varphi + \frac{n}{2}C(X,Y)\cdot\varphi - \frac{n}{2}\frac{1}{(n-2)(n-1)}\left(Y^\flat\wedge\nabla_X dA - X^\flat\wedge\nabla_Y dA\right)\\
&- \frac{n}{2(n-1)}\left(g(\nabla_X dA,Y) - g(\nabla_Y dA,X)\right)\cdot\varphi - \big(\frac{1}{(n-2)(n-1)}X^\flat\wedge Y^\flat\wedge dA + \frac{n-3}{2(n-1)}dA(X,Y)\\
&+ \frac{1}{n-2}((X\lrcorner dA)\wedge Y^\flat - (Y\lrcorner dA)\wedge X^\flat))\cdot D^A\varphi.
\end{aligned}
$$

Remark 7.13 For Riemannian 4-manifolds these integrability conditions have already appeared in [CM13].

We now clarify the relation of CCKS to conformal Killing forms. For this purpose, we introduce the following set of differential forms for a spinor field $\varphi\in\Gamma(S^g)$ and $k\in\mathbb{N}$:

$$
\begin{aligned}
g\left(\alpha^k_\varphi,\alpha\right) &:= d_k\cdot\langle\alpha\cdot\varphi,\varphi\rangle_{S^g}, & \alpha\in\Omega^k(M),\\
g\left(\alpha^{k+1}_0,\beta\right) &:= \frac{2d_k(-1)^{k-1}}{n}h\left(\langle\beta\cdot D^A\varphi,\varphi\rangle_{S^g}\right), & \beta\in\Omega^{k+1}(M),\\
g\left(\alpha^{k-1}_\mp,\gamma\right) &:= \frac{2d_k(-1)^{k-1}}{n}h\left(\langle\gamma\cdot D^A\varphi,\varphi\rangle_{S^g}\right), & \gamma\in\Omega^{k-1}(M),
\end{aligned}
\tag{7.17}
$$

where $h(z) := \frac{1}{2}\left(z + (-1)^{k(p+1+\frac{k-1}{2})}\overline{z}\right)$. $d_k\in U(1)$ are powers of i, ensuring that α^k_φ is indeed a real form. A straightforward calculation using only the $Spin^c$-twistor equation yields that for $\varphi\in\ker P^A$:

$$
\nabla^g_X\alpha^k_\varphi = X\lrcorner\alpha^{k+1}_0 + X^\flat\wedge\alpha^{k-1}_\mp, \tag{7.18}
$$

i.e. α^k_φ is a conformal Killing form. Such forms have been studied intensively in [Sem01, Lei05]. From (7.18) we deduce that $(k+1)\alpha^{k+1}_0 = d\alpha^k_\varphi$ and $(n-k+1)\alpha^{k-1}_\mp = d^*\alpha^k_\varphi$. Moreover, in case $k=1$ (7.18) is equivalent to say that $V_\varphi = \left(\alpha^1_\varphi\right)^\sharp$ is a conformal vector field. Note that under a conformal change of the metric with factor $e^{2\sigma}$, α^k_φ transforms with factor $e^{(k+1)\sigma}$, and thus V_φ depends on the conformal class only.

We now derive further equations for the *Lorentzian* case[1] and $k=1$. Note that in this case we may set $d_1 = 1$. Let us introduce further forms for $\varphi\in\Gamma(S^g)$ by setting

$$
\begin{aligned}
g\left(\alpha^j_{dA},\alpha\right) &:= \frac{1}{(n-2)(n-1)}\cdot\mathrm{Re}\,\langle dA\cdot\varphi,\alpha\cdot\varphi\rangle_{S^g}, & \alpha\in\Omega^j(M),\\
g\left(\widetilde{\alpha}^2_0,\beta\right) &:= \frac{2}{n}\mathrm{Im}\,\langle\beta\cdot D^A\varphi,\varphi\rangle_{S^g}, & \beta\in\Omega^2(M),\\
\widetilde{\alpha}_\mp &:= \frac{2}{n}\mathrm{Im}\,\langle D^A\varphi,\varphi\rangle_{S^g}.
\end{aligned}
$$

[1] In fact, all the following equations can be obtained in arbitrary signatures where one has to change some signs and real and imaginary parts.

Differentiating the various forms and straightforward calculation reveals that the twistor equation and (7.10) yield the following system of equations:

$$\begin{pmatrix} \nabla^g_X & -X\lrcorner & -X^\flat\wedge & 0 \\ -K^g(X)\wedge & \nabla^g_X & 0 & X^\flat\wedge \\ -K^g(X)\lrcorner & 0 & \nabla^g_X & -X\lrcorner \\ 0 & K^g(X)\lrcorner & -K^g(X)\wedge & \nabla^g_X \end{pmatrix}\begin{pmatrix} \alpha^1_\varphi \\ \alpha^2_0 \\ \alpha_\mp \\ \frac{2}{n^2}\alpha^1_{D^A\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{n-2}(X\lrcorner\frac{1}{i}dA)^\sharp\lrcorner\alpha^3_\varphi - X^\flat\wedge\alpha^1_{dA} + X\lrcorner\alpha^3_{dA} \\ X\lrcorner\alpha^1_{dA} \\ \frac{1}{n-1}\left(\frac{1}{n-2}(X\lrcorner\frac{1}{i}dA)^\sharp\lrcorner\widetilde{\alpha}^2_0 + \widetilde{\alpha}_\mp\cdot(X\lrcorner\frac{1}{i}dA)\right) \end{pmatrix} \tag{7.19}$$

Remark 7.14 Elements in the kernel of the operator on the left hand side define precisely the *normal* conformal Killing forms resp. vector fields which have appeared in the second chapter . As remarked before, for a conformal vector field V, $V^\flat$ being normal conformal is equivalent to the curvature conditions (see [Gov06, GS08])

$$V\lrcorner W^g = 0,\ V\lrcorner C^g = 0, \text{i.e. } C^g(V,\cdot) = 0 \in \mathfrak{X}(M).$$

Due to the dA−terms, the associated vector to a CCKS is in general *no* normal conformal vector field, in contrast to the *Spin* setting. In general, there is no additional equation for α^1_φ only, except the conformal Killing equation.

We next study the relation of V_φ with the two main curvature quantities related to a CCKS, namely W^g and dA. As before, we will restrict ourselves to the Lorentzian case. First, we show that V_φ preserves dA.

Proposition 7.15 *It holds that* $V_\varphi\lrcorner\left(\frac{1}{i}dA\right) = \frac{2(1-n)}{n}d\left(Im\langle D^A\varphi,\varphi\rangle_{S^g}\right)$. *In particular, we have that*

$$L_{V_\varphi}\frac{1}{i}dA = 0.$$

Proof. Let us write $\omega = \frac{1}{i}dA \in \Omega^2(M)$. We have for $Y \in TM$:

$$\begin{aligned}
(V_\varphi\lrcorner\omega)(Y) =& \omega(V_\varphi,Y) = -\omega(Y,V_\varphi) = -g\left((Y\lrcorner\omega)^\sharp, V_\varphi\right) \\
=& -\langle (Y\lrcorner\omega)\cdot\varphi,\varphi\rangle_{S^g} \\
=& \frac{1}{i}\frac{2(1-n)}{n}\cdot\langle\nabla^A_Y D^A\varphi,\varphi\rangle_{S^g} + (n-1)\cdot\underbrace{\frac{1}{i}\langle K^g(Y)\cdot\varphi,\varphi\rangle_{S^g}}_{\in i\mathbb{R}} \\
&+ \frac{1}{n-2}\cdot\underbrace{\langle (Y^\flat\wedge\omega)\cdot\varphi,\varphi\rangle_{S^g}}_{\in i\mathbb{R}} \in \mathbb{R}\ ((7.10)\text{ inserted}) \\
=& 2\frac{1-n}{n}\cdot\mathrm{Im}\langle\nabla^A_Y D^A\varphi,\varphi\rangle_{S^g} = 2\frac{1-n}{n}\cdot\mathrm{Im}(Y(\langle D^A\varphi,\varphi\rangle_{S^g}) + \frac{1}{n}\underbrace{\langle Y\cdot D^A\varphi, D^A\varphi\rangle_{S^g}}_{\in\mathbb{R}}) \\
=& 2\frac{1-n}{n}\cdot d\left(\mathrm{Im}\ \langle D^A\varphi,\varphi\rangle_{S^g}\right)(Y).
\end{aligned}$$

The second formula follows directly with Cartans relation $L = \lrcorner\circ d + d\circ\lrcorner$. □

Remark 7.16 For 4-dimensional Lorentzian manifolds an alternative proof of Proposition 7.15 is given in [CKM+14].

Next, we investigate how V_φ inserts into the Weyl tensor. We have by definition and using (7.15)for $X,Y,Z \in TM$:

$$\begin{aligned} W^g(V_\varphi, X, Y, Z) &= -\langle \varphi, W^g(X,Y,Z) \cdot \varphi \rangle_{S^g} \\ &= \langle \varphi, Z \cdot W^g(X,Y) \cdot \varphi \rangle_{S^g} - \langle \varphi, (Z^\flat \wedge W^g(X,Y)) \cdot \varphi \rangle_{S^g} \in \mathbb{R} \end{aligned} \tag{7.20}$$

In Lorentzian signature, $\langle \varphi, \omega \cdot \varphi \rangle_{S^g} \in i\mathbb{R}$ for $\omega \in \Omega^3(M)$. Inserting the integrability condition (7.16) and keeping only real terms, we arrive with the aid of (7.15) at

$$W^g(V_\varphi, X, Y, Z) = c_n \cdot \langle \varphi, \Big(Z \lrcorner (X^\flat \wedge Y^\flat \wedge dA) + \frac{3-n}{2} Z^\flat \wedge (X^\flat \wedge (Y \lrcorner dA) - Y^\flat \wedge (X \lrcorner dA)) \Big) \cdot \varphi \rangle_{S^g},$$

where $c_n = -\frac{2}{(n-2)(n-1)}$. By permuting X, Y and Z, it is pure linear algebra to conclude that the last expression vanishes for all $X,Y,Z \in TM$ if and only if $\langle (X^\flat \wedge Y^\flat \wedge (Z \lrcorner dA)) \cdot \varphi, \varphi \rangle_{S^g} = 0$ for all $X,Y,Z \in TM$. We can express this as follows:

Proposition 7.17 *For a Lorentzian CCKS $\varphi \in \ker P^A$, it holds the curvature relation*

$$V_\varphi \lrcorner W^g = 0 \Leftrightarrow (Z \lrcorner \frac{1}{i} dA)^\sharp \lrcorner \alpha^3_\varphi = 0 \;\; \forall Z \in TM.$$

In particular, one does not need to compute W^g to check the first condition for V_φ being normal conformal. One obtains another relation between dA and V_φ by requiring the imaginary part of (7.20) to vanish. Again, inserting (7.16) and straightforward manipulations yield that[2]

$$\begin{aligned} 0 =& (3-n)(g(V_\varphi, Z) dA(X,Y) + g(V_\varphi, X) dA(Y,Z) + g(V_\varphi, Y) dA(Z,X) - g(X,Z) g((Y \lrcorner dA)^\sharp, V_\varphi) \\ &+ g(Y,Z) g((X \lrcorner dA)^\sharp, V_\varphi)) + \frac{2}{n-2} g\left(\alpha^5_\varphi, X^\flat \wedge Y^\flat \wedge Z^\flat \wedge dA\right) + i(n-1) \cdot g\left(\alpha^3_\varphi, Z^\flat \wedge W^g(X,Y)\right) \end{aligned}$$

As a consistency check, note that all integrability conditions including the Weyl curvature become trivial in case $n = 3$. Finally, inserting (7.16) into $g\left(\alpha^2_\varphi, W^g(X,Y)\right) = i \cdot \langle \varphi, W^g(X,Y) \cdot \varphi \rangle_{S^g} \in \mathbb{R}$ and splitting into real and imaginary part, we arrive at the relations

$$\begin{aligned} i \cdot (1-n) g\left(\alpha^2_\varphi, W^g(X,Y)\right) &= (3-n) dA(X,Y) \langle \varphi, \varphi \rangle_{S^g} + \frac{2}{n-2} g\left(\alpha^4_\varphi, X^\flat \wedge Y^\flat \wedge dA\right), \\ 0 &= \langle \varphi, \left(X^\flat \wedge (Y \lrcorner dA) - Y^\flat \wedge (X \lrcorner dA)\right) \cdot \varphi \rangle_{S^g}. \end{aligned}$$

We conclude these general observations about CCKS with some remarks regarding the zero set $Z_\varphi \subset M$ of a CCKS $\varphi \in \ker P^A$. By (7.10) every $x \in Z_\varphi$ satisfies $\nabla D^A \varphi(x) = 0$. This observation allows one to prove literally as in [BFGK91] and [Lei01] the following:

Proposition 7.18 *Let $\varphi \in \ker P^A$ be a CCKS on $(M^{p,q}, g)$. If $\gamma : I \to Z_\varphi \subset M$ is a curve which runs in the zero set, then γ is isotropic. If $p = 0$, then Z_φ consists of a countable union of isolated points. If $p = 1$, then the image of every geodesic γ_v starting in $x \in Z_\varphi$ with initial velocity v satisfying that $v \cdot D^g \varphi(x) = 0$ is contained in Z_φ*

This ends our discussion of general properties of the CCKS-equation and its relations to curvature. We next turn to construction principles, classification results and relations to special geometries in small dimensions.

[2] In the following $g((X \lrcorner dA)^\sharp, Y) := i \cdot g((X \lrcorner \frac{1}{i} dA)^\sharp, Y) \in i\mathbb{R}$ for $X, Y \in TM$.

7.4 Lorentzian CCKS and CR-geometry

The Fefferman metric

The purpose of this section is to give a global construction principle of CCKS with nontrivial curvature $dA \in \Omega^2(M, i\mathbb{R})$ on Lorentzian manifolds $(M^{1,2n+1}, g)$ starting from $(2n+1)$−dimensional strictly pseudoconvex structures. This can be viewed as the $Spin^c$-analogue of [Bau99], and in fact the construction is quite similar. In view of the subsequent considerations, let us review the following well-known fact:

Consider a pseudo-Riemannian Kähler manifold $(M^{p,q}, g, J)$, where $(p,q) = (2p', 2q')$, $p + q = 2n$, endowed with its canonical $Spin^c$−structure (cf. Remark 7.4), where the $U(p', q')$−reduction $\mathcal{P}_U$ of $\mathcal{P}^g$ is given by considering only pseudo-orthonormal bases of the form $(s_1, J(s_1), ..., s_n, J(s_n))$. As J is parallel, ∇^g reduces to a connection $\omega^g_U \in \Omega^1(\mathcal{P}_U, \mathfrak{u}(p', q'))$. By Remark 7.4, $\mathcal{P}_U$ and the S^1-bundle $\mathcal{P}_1$ are related by det-reduction,

$$\phi_1 : \mathcal{P}_U \to \mathcal{P}_1 = \mathcal{P}_U \times_{\det} S^1.$$

Whence there exists a connection $A \in \Omega^1(\mathcal{P}_1, i\mathbb{R})$, uniquely determined by

$$(\phi_1(s))^* A =: A^{\phi_1 \circ s} = \operatorname{tr}\left(\omega^g_U\right)^s \text{ for } s \in \Gamma(V, \mathcal{P}_U).$$

One calculates that $dA(X,Y) = i \cdot Ric^g(X, JY)$.

Proposition 7.19 *On every pseudo-Riemannian Kähler manifold $(M^{p,q}, g, J)$ there exists a ∇^A- parallel spinor.*

Proof. As known from [Kir86] the complex spinor module $\Delta^{\mathbb{C}}_{2n}$ decomposes into $\Delta^{\mathbb{C}}_{2n} = \oplus^n_{k=0} \Delta^{k,\mathbb{C}}_{2n}$, where the $\Delta^{k,\mathbb{C}}_{2n}$ are eigenspaces of the action of the Kähler form $\Omega = \langle \cdot, J \cdot \rangle_{p,q}$ to the eigenvalue $\mu_k = (n-2k)i$. $\Delta^{n,\mathbb{C}}_{2n}$ turns out to be one-dimensional, in the notation from Remark 7.1 it is spanned by $u(-1, ..., -1)$ and acted on trivially by $U(p', q')$, i.e.

$$l(U) \cdot u(-1, ..., -1) = u(-1, ..., -1) \text{ for } U \in U(p', q'). \tag{7.21}$$

We define a global section $\varphi \in \Gamma(M, S^g)$ by $\varphi_{|V} := [\phi_c(s), u(-1, ..., -1)]$ for $s \in \Gamma(V, \mathcal{P}_U)$, where ϕ_c is given in Remark 7.4. (7.21) yields that this is well-defined, i.e. independent of the chosen s. Writing $s^* \omega^g_U$ and $(\phi_1(s))^* A$ in terms of ∇^g is straightforward and then one directly calculates with the local formula (7.4) that $\nabla^A \varphi = 0$. □

The rest of this section is devoted to the conformal analogue of this construction. We closely follow [Bau99] and sometimes refer to this article when leaving out steps which are identical in our construction. To start with, let us recall some basic facts from CR-geometry[3]:

Definition 7.20 *Let M^{2n+1} be a smooth, connected, oriented manifold of odd dimension $2n+1$. A* real CR-structure *on M is a pair (H, J), where*

1. *$H \subset TM$ is a real $2n$−dimensional subbundle,*

[3] Here we only work in the picture of real CR structures. Our notation regarding CR-geometry follows [Bau99, BJ10, Sta11]. We refer to these references for further details.

2. *$J : H \to H$ is an almost complex structure on H: $J^2 = -Id_H$,*

3. *If $X, Y \in \Gamma(H)$, then $[JX, Y] + [X, JY] \in \Gamma(H)$ and the integrability condition $N_J(X, Y) := J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] \equiv 0$ holds.*

(M, H, J) is called a (oriented) CR-manifold. We fix a nowhere vanishing 1-form $\theta \in \Omega^1(M)$ with $\theta_{|H} \equiv 0$, which exists as M is oriented and is unique up to multiplication with a nowhere vanishing function. With this choice we define the Levi-form L_θ on H as

$$L_\theta(X, Y) := d\theta(X, JY)$$

for $X, Y \in \Gamma(H)$. (M, H, J, θ) is called a strictly-pseudoconvex pseudo-Hermitian manifold if L_θ is positive definite. In this case, θ is a contact form and we let T denote the characteristic vector field of the contact form θ, i.e. $\theta(T) \equiv 1$ and $T \lrcorner d\theta \equiv 0$.

It is a standard fact that under the above assumptions $g_\theta := L_\theta + \theta \circ \theta$ defines a Riemannian metric on M. Clearly, the $SO^+(2n+1)-$frame bundle $\mathcal{P}_M^{g_\theta}$ reduces to the $U(n)$ bundle

$$\mathcal{P}_{U,H} := \{(X_1, JX_1, ..., X_n, JX_n, T) \mid (X_1, JX_1, ..., X_n, JX_n) \text{ pos. oriented ONB of } (H, L_\theta)\},$$

where $U(n) \hookrightarrow SO^+(2n) \hookrightarrow SO^+(2n+1)$. By Remark 7.4 this induces a $Spin^c(2n+1)$-structure $(\mathcal{Q}_M^c = \mathcal{P}_{U,H} \times_l Spin^c(2n+1), f_M^c)$ on (M, g_θ), where $Spin^c(2n) \hookrightarrow Spin^c(2n+1)$, with auxiliary bundle $\mathcal{P}_{1,M} = \mathcal{P}_{U,H} \times_{\det} S^1$ and natural reduction maps

$$\phi_{c,M} : \mathcal{P}_{U,H} \to \mathcal{Q}_M^c,\ \phi_{1,M} : \mathcal{P}_{U,H} \to \mathcal{P}_{1,M}.$$

There is a special covariant derivative on a strictly pseudoconvex manifold, the Tanaka Webster connection $\nabla^W : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$, uniquely determined by requiring it to be metric wrt. g_θ and the torsion tensor Tor^W to satisfy for $X, Y \in \Gamma(H)$

$$\begin{aligned} \mathrm{Tor}^W(X, Y) &= L_\theta(JX, Y) \cdot T, \\ \mathrm{Tor}^W(T, X) &= -\frac{1}{2}([T, X] + J[T, JX]). \end{aligned}$$

Let $\mathrm{Ric}^W \in \Omega^2(M, i\mathbb{R})$ and $R^W \in C^\infty(M, \mathbb{R})$ denote the Tanaka-Webster Ricci-and scalar curvature (see [Bau99]). As $\nabla^W g_\theta = 0, \nabla^W T = 0$ and $\nabla^W J = 0$, it follows that ∇^W descends to a connection $\omega^W \in \Omega^1(\mathcal{P}_{U,H}, \mathfrak{u}(n))$. In the standard way, this induces a connection $A^W \in \Omega^1(\mathcal{P}_{1,M}, i\mathbb{R})$, uniquely determined by[4]

$$(\phi_{1,M}(s))^* A^W = \mathrm{Tr}\left(s^* \omega^W\right),$$

where $s \in \Gamma(V, \mathcal{P}_{U,H})$ is a local section. Two connections on a S^1-bundle over M differ by an element of $\Omega^1(M, i\mathbb{R})$. Consequently,

$$A_\theta := A^W + \frac{i}{2(n+1)} R^W \theta$$

[4]Note that this sign differs from the one in the construction in [Bau99]! We use a different realisation of the canonical line bundle.

is a S^1–connection on $\mathcal{P}_{1,M}$. Let $\pi : \mathcal{P}_{1,M} \to M$ denote the projection. Setting

$$h_\theta := \pi^* L_\theta - i\frac{4}{n+2}\pi^*\theta \circ A_\theta$$

defines a right-invariant Lorentzian metric on the total space $F := \mathcal{P}_{1,M}$ considered as manifold, the so-called Fefferman metric. Its further properties are discussed in [Lei07, Gra87]. In particular, one finds that the conformal class $[h_\theta]$ does not depend on θ, which is unique up to multiplication with a nowhere vanishing function, but on the CR-data (M, H, J) only.
In the next section we define a natural $Spin^c(1,2n+1)$-structure on the Lorentzian manifold (F, h_θ) and show that it admits a CCKS for a natural choice of A.

$Spin^c$–characterization of Fefferman spaces

This subsection is mainly an application of the spinor calculus for S^1-bundles with isotropic fibres over strictly pseudoconvex spin manifolds from [Bau99] to our case with slight modifications as we are dealing with $Spin^c$-structures. Let (F, h_θ) denote the Fefferman space of (M, H, J, θ), where $F = \mathcal{P}_{1,M} \xrightarrow{\pi} M$ is the S^1-bundle. Let $N \in \mathfrak{X}(F)$ denote the fundamental vector field of F defined by $\frac{n+2}{2}i \in i\mathbb{R}$, i.e. $N(f) := \frac{d}{dt}_{|t=0}\left(f \cdot e^{\frac{n+2}{2}it}\right)$ for $f \in F$. For a vector field $X \in \mathfrak{X}(M)$, let $X^* \in \mathfrak{X}(F)$ be its A_θ–horizontal lift. We define the h_θ-orthogonal timelike and spacelike vectors $s_1 := \frac{1}{\sqrt{2}}(N - T^*)$, $s_2 := \frac{1}{\sqrt{2}}(N + T^*)$ which are of unit length. Let the time orientation of (F, h_θ) be given by s_1 and the space orientation by vectors $(s_2, X_1^*, JX_1^*, ..., X_n^*, JX_n^*)$, where $(X_1, JX_1, ..., X_n, JX_n, T) \in \mathcal{P}_{U,H}$. Obviously, the bundle

$$\mathcal{P}_{U,F} := \{(s_1, s_2, X_1^*, JX_1^*, ..., X_n^*, JX_n^*) \mid (X_1, JX_1, ..., X_n, JX_n, T) \in \mathcal{P}_{U,H}\} \to F$$

is a $U(n) \hookrightarrow SO^+(1, 2n+1)$ reduction of the orthonormal frame bundle $\mathcal{P}_F^{h_\theta} \to F$ and $\mathcal{P}_{U,F} \cong \pi^*\mathcal{P}_{U,H}$. It follows again with Remark 7.4 that there is a canonically induced $Spin^c(1, 2n+1)$-structure for (F, h_θ), namely

$$\left(\mathcal{Q}_F^c := \mathcal{P}_{U,F} \times_l Spin^c(1, 2n+1), f_F^c, \mathcal{P}_{1,F} := \mathcal{P}_{U,F} \times_{\det} S^1\right),$$

where $U(n) \xrightarrow{l} Spin^c(2n) \hookrightarrow Spin^c(1, 2n+1)$, together with reduction maps

$$\phi_{c,F} : \mathcal{P}_{U,F} \to \mathcal{Q}_F^c,\ \phi_{1,F} : \mathcal{P}_{U,F} \to \mathcal{P}_{1,F}.$$

There are two distinct natural maps between the S^1-bundles $\mathcal{P}_{1,F}$ and F: Viewing $\mathcal{P}_{1,F}$ as the total space of an S^1–bundle over the manifold F gives the projection $\pi_F : \mathcal{P}_{1,F} \to F$, whereas the isomorphism $\pi^*\mathcal{P}_{U,H} \cong \mathcal{P}_{U,F}$ leads to a natural S^1–equivariant bundle map

$$\begin{aligned}\widehat{\pi}_F : \mathcal{P}_{1,F} \cong \pi^*\mathcal{P}_{U,H} \times_{\det} S^1 &\to F \cong \mathcal{P}_{U,H} \times_{\det} S^1,\\ [v, z] &\mapsto [\pi_U(v), z],\end{aligned}$$

with $\pi_U : \pi^*\mathcal{P}_{U,H} \to \mathcal{P}_{U,H}$ being the natural projection.
The proof of the following statements is a matter of unwinding the definitions:

Proposition 7.21 *Let $s \in \Gamma(V, \mathcal{P}_{U,H})$ be a local section for some open set $V \subset M$ and define $\widehat{s} \in \Gamma(\pi^{-1}(V), \mathcal{P}_{U,F} \cong \pi^*\mathcal{P}_{U,H})$ by $\widehat{s}(f) := (f, s(\pi(f))) \in (\pi^*\mathcal{P}_{U,H})_f$. Further, let $\pi_U : \pi^*\mathcal{P}_{U,H} \to \mathcal{P}_{U,H}$ be the natural projection. Then the following diagram commutes:*

$$\begin{tikzcd} F \arrow[r, "\widehat{s}"] \arrow[d, "\pi"] & \mathcal{P}_{U,F} \arrow[r, "\phi_{1,F}"] \arrow[d, "\pi_U"] & \mathcal{P}_{1,F} \arrow[d, "\widehat{\pi}_F"] \\ M \arrow[r, "s"] & \mathcal{P}_{U,H} \arrow[r, "\phi_{1,M}"] & F = \mathcal{P}_{1,M} \end{tikzcd}$$

Proposition 7.22 *Let $A \in \Omega^1(F, i\mathbb{R})$ be a connection on the S^1-bundle $F = \mathcal{P}_{1,M} \overset{\pi}{\to} M$. Then $\widehat{\pi}_F^* A \in \Omega^1(\mathcal{P}_{1,F}, i\mathbb{R})$ is a connection on the S^1-bundle $\mathcal{P}_{1,F} \overset{\pi_F}{\to} F$. Locally, A and $\widehat{\pi}_F^* A$ are related as follows: Let $s \in \Gamma(V, \mathcal{P}_{U,H})$ and let $\widehat{s} \in \Gamma(\pi^{-1}(V), \mathcal{P}_{U,F})$ be the induced local section as in Proposition 7.21. It holds that*

$$(\widehat{\pi}_F^* A)^{\phi_{1,F}(\widehat{s})} = \pi^*\left(A^{\phi_{1,M}(s)}\right) \in \Omega^1(\pi^{-1}(V), i\mathbb{R}).$$

Let us now turn to spinor fields on F. By construction, the $Spin^c(2n+1)$-bundle $\mathcal{Q}_M^c \to M$ reduces to the $Spin^c(2n)$-bundle $\mathcal{Q}_H^c := \mathcal{P}_{U,H} \times_l Spin^c(2n) \to M$. We introduce the reduced spinor bundle of M,

$$S_H := S_H^{g_\theta} := \mathcal{Q}_H^c \times_{\Phi_{2n}} \Delta_{2n}^{\mathbb{C}} \cong \mathcal{P}_{U,H} \times_{\Phi_{2n} \circ l} \Delta_{2n}^{\mathbb{C}}.$$

This allows us to express the spinor bundle $S_F := S_F^{h_\theta} \to F$ as

$$\begin{aligned} S_F &= \mathcal{Q}_F^c \times_{\Phi_{1,2n+1}} \Delta_{1,2n+1}^{\mathbb{C}} \cong \pi^*\mathcal{P}_{U,H} \times_{\Phi_{1,2n+1} \circ l} \Delta_{2n+1}^{\mathbb{C}} \\ &\cong \pi^* S_H \oplus \pi^* S_H. \end{aligned} \tag{7.22}$$

The second step is purely algebraic and follows from the decomposition of $\Delta_{1,2n+1}^{\mathbb{C}}$ into the sum $\Delta_{2n}^{\mathbb{C}} \oplus \Delta_{2n}^{\mathbb{C}}$ of $Spin^c(2n) \hookrightarrow Spin^c(1, 2n+1)$-representations as presented in [Bau99], where $\mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n+2}$ via $x \mapsto (0, 0, x)$. Under the identification (7.22) we have (cf. [Bau99], Proposition 18)

$$\begin{aligned} s_1 \cdot (\varphi, \phi) &= (-\phi, -\varphi), \\ s_2 \cdot (\varphi, \phi) &= (-\phi, \varphi), \\ X^* \cdot (\varphi, \phi) &= (-X \cdot \varphi, X \cdot \phi). \end{aligned} \tag{7.23}$$

This identification allows us to define a global section in $\pi^* S_H \oplus (F \times \{0\}) \subset S_F$ in analogy to the Kähler case: $u(-1, ..., -1) \in \Delta_{2n}^{\mathbb{C}}$ is the (up to S^1-action) unique unit-norm spinor in the eigenspace of the Kähler form on $\mathbb{R}^{2n}$ to the eigenvalue $-i \cdot n$. Let $s : V \subset M \to P_{U,H}$ be a local section. We set

$$\varphi(p) := [\phi_{c,F}(\widehat{s}(p)), u(-1, ..., -1)], \ p \in \pi^{-1}(V).$$

By (7.21) this is independent of the choice of s. Thus, $\varphi \in \Gamma(F, S_F)$ is defined. As last ingredient we introduce the connection

$$A := \widehat{\pi}_F^* A^W + A^W \in \Omega^1(\mathcal{P}_{1,F}, i\mathbb{R})$$

on the S^1-bundle $\mathcal{P}_{1,F} \overset{\pi_F}{\to} F$ [5].

[5] This is to be read as follows: $\widehat{\pi}_F^* A^W$ is a connection on $\mathcal{P}_{1,F}$ by Proposition 7.22. Any other connection is obtained by adding an element of $\Omega^1(F, i\mathbb{R})$, which we choose to be the connection A^W here, i.e. $A = \widehat{\pi}_F^* A^W + \pi_F^* A^W$.

Theorem 7.23 *The spinor field $\varphi \in \Gamma(F, S_F^{h_\theta})$ is a CCKS wrt. A, i.e. $\varphi \in ker\ P^{A,h_\theta}$. The curvature $dA \in \Omega^2(F, i\mathbb{R})$ is given by*

$$dA = 2\pi^*_{F\to M} Ric^W.$$

In particular, φ descends to a twistor spinor on a spin manifold iff the Tanaka Webster connection is Ricci-flat. The associated vector field V_φ satisfies

1. *V_φ is a regular isotropic Killing vector field,*
2. $\nabla^A_{V_\varphi}\varphi = \frac{1}{\sqrt{2}} i\varphi$,
3. *V_φ is normal, i.e. $V_\varphi \lrcorner W^{h_\theta} = 0, V_\varphi \lrcorner C^{h_\theta} = 0$,*
   Furthermore, $K^{h_\theta}(V_\varphi, V_\varphi) = const. < 0$,
4. $V_\varphi \lrcorner dA = 0$.

Proof. Applying the local formula (7.4) to φ and using Proposition 7.22, we find for a local section $s = (X_1, ..., X_{2n}, T) \in \Gamma(V, \mathcal{P}_{U,H})$ and a vector $Y \in \Gamma(\pi^{-1}(V), TF)$ that

$$\begin{aligned}\nabla^A_Y \varphi_{|\pi^{-1}(V)} = &-\frac{1}{2} h_\theta(\nabla^{h_\theta}_Y s_1, s_2) s_1 \cdot s_2 \cdot \varphi - \frac{1}{2}\sum_{k=1}^{2n} h_\theta(\nabla^{h_\theta}_Y s_1, X_k^*) s_1 \cdot X_k^* \cdot \varphi \\ &+ \frac{1}{2}\sum_{k=1}^{2n} h_\theta(\nabla^{h_\theta}_Y s_2, X_k^*) s_2 \cdot X_k^* \cdot \varphi + \frac{1}{2}\sum_{k<l} h_\theta(\nabla^{h_\theta}_Y X_k^*, X_l^*) X_k^* \cdot X_l^* \cdot \varphi \\ &+ \frac{1}{2}\left(A^W\right)^{\phi_{1,M}(s)}(d\pi(Y)) \cdot \varphi + \frac{1}{2} A^W(Y) \cdot \varphi,\end{aligned}$$

where for $X \in \mathfrak{X}(M)$, the vector field $X^* \in \mathfrak{X}(F)$ is the horizontal lift wrt. A_θ (not A^W!). The calculation of the local connection 1-forms of ∇^{h_θ} and their pointwise action on the spinor $u(-1, ..., -1)$ has been carried out in [Bau99]. Inserting these expressions and taking into account the slight differences to our construction[6] we arrive at

$$\begin{aligned}\nabla^A_N \varphi_{|\pi^{-1}(V)} &= \left(-i\frac{n}{4}\cdot\varphi + \frac{1}{2} A^W(N)\cdot\varphi, 0\right), \\ \nabla^A_{T^*}\varphi_{|\pi^{-1}(V)} &= \left(i\frac{R^W}{4(n+1)}\varphi - \frac{1}{2}\mathrm{Tr}\ \omega_s(T) + \frac{1}{2}\left(\left(A^W\right)^{\phi_{1,M}(s)}(T) + A^W(T^*)\right)\cdot\varphi, 0\right), \\ \nabla^A_{X^*}\varphi_{|\pi^{-1}(V)} &= \left(-\frac{1}{2}\mathrm{Tr}\ \omega_s(T) + \frac{1}{2}\left(\left(A^W\right)^{\phi_{1,M}(s)}(X) + A^W(X^*)\right)\cdot\varphi, 0\right) - \frac{1}{4}(X\lrcorner d\theta)^* \cdot T^* \cdot \varphi.\end{aligned}$$

Here, $\omega_s := s^* A^W \in \Omega^1(V, \mathfrak{u}(n))$. By definition, we have that

$$\begin{aligned}A^W(N) &= i\cdot\frac{n+2}{2}, \\ \left(A^W\right)^{\phi_{1,M}(s)}(T) + A^W(T^*) &= \mathrm{Tr}\ \omega_s(T) - i\frac{R^W}{2(n+1)}, \\ \left(A^W\right)^{\phi_{1,M}(s)}(X) + A^W(X^*) &= \mathrm{Tr}\ \omega_s(X).\end{aligned}$$

[6] Concretely, in [Bau99] the induced Webster connection on the line bundle is defined with a different sign which changes the sign of its curvature. Moreover, in [Bau99] the Fefferman *spin* metric comes with a factor $\frac{8}{n+2}$ instead of $\frac{4}{n+2}$

Inserting this and noticing that for $X \in \{X_1, ..., X_{2n}\}$ the 1-form $X \lrcorner d\theta$ acts on the spinor bundle by Clifford multiplication with $J(X)$, we arrive at

$$\begin{aligned}
\nabla^A_N \varphi &= \frac{1}{2} i\varphi, \\
\nabla^A_{T^*} \varphi &= 0, \\
\nabla^A_{X^*} \varphi &= \left(0, -\frac{\sqrt{2}}{4} J(X) \cdot \varphi\right).
\end{aligned}$$

Using (7.23), we conclude that

$$-s_1 \cdot \nabla^A_{s_1} \varphi = s_2 \cdot \nabla^A_{s_2} \varphi = X^* \cdot \nabla^A_{X^*} \varphi = \left(0, \frac{1}{2\sqrt{2}} i\varphi\right),$$

where $X \in \{X_1, ..., X_{2n}\}$. This shows that $h(Y,Y) Y \cdot \nabla^A_Y \varphi$ is independent of the vector Y with length ± 1, i.e. $\varphi \in \ker P^A$. It is straightforward to calculate

$$\begin{aligned}
V_\varphi &= -\langle s_1 \cdot \varphi, \varphi \rangle_{S^{h_\theta}} s_1 + \langle s_2 \cdot \varphi, \varphi \rangle_{S^{h_\theta}} s_2 + \sum_{k=1}^{2n} \langle X_k^* \cdot \varphi, \varphi \rangle_{S^{h_\theta}} X_k^* \\
&= -s_1 - s_2 = -\sqrt{2} N.
\end{aligned}$$

We conclude that V_φ is regular and isotropic. It is shown in [Bau99], Proposition 19 that the vertical vector field N is Killing. The claimed relation of V_φ to the curvature tensors of h_θ is true for any fundamental vertical vector field on a Fefferman space (see [Gra87]).

It remains to calculate dA and to prove 4.: Let $s \in \Gamma(V, \mathcal{P}_{U,H})$. It holds that (cf. [Bau99]) $dA^W = \text{Tr } d\omega_s = Ric^W \in \Omega^2(M, i\mathbb{R})$. Considered as a 2-form on F, the curvature dA is thus using Proposition 7.22 given by

$$\begin{aligned}
dA &= d\left(\widehat{\pi}_F^* A^W\right)^{\phi_{1,F}(\widehat{s})} + \pi^* dA^W = \pi^* d\text{Tr } \omega_s + \pi^* Ric^W \\
&= 2\pi^* Ric^W
\end{aligned}$$

As dA is the lift of a 2-form on M, it follows immediately that the fundamental vector field $V_\varphi = -\sqrt{2}N$ inserts trivially into dA. □

Remark 7.24 Generically, we find only one CCKS on the Fefferman space. One can define another natural global section in S_F in analogy to the spin case in [Bau99]. However, there is in general no S^1–connection which turns it into a CCKS. This is in complete analogy to the Kähler case: On a Kähler manifold there is a second natural global section in the spinor bundle constructed out of the eigenspinor to the other extremal eigenvalue of the Kähler form on spinors which in general is no $Spin^c$-parallel spinor (cf. [Mor97]).

As in the $Spin$–case we can also prove a converse of the last statement:

Theorem 7.25 *Let $(B^{1,2n+1}, h)$ be a Lorentzian $Spin^c$-manifold. Let $A \in \Omega^1(\mathcal{P}_1, i\mathbb{R})$ be a connection on the underlying S^1-bundle and let $\varphi \in \Gamma(S^g)$ be a nontrivial CCKS wrt. A such that*

1. *The Dirac current $V := V_\varphi$ of φ is a regular isotropic Killing vector field*[7]*,*
2. $V \lrcorner W^h = 0$ *and* $V \lrcorner C^h = 0$*, i.e. V is a normal conformal vector field,*
3. $V \lrcorner dA = 0$,
4. $\nabla^A_V \varphi = ic\varphi$*, where $c = const \in \mathbb{R}\backslash\{0\}$.*

Then (B,h) is a S^1-bundle over a strictly pseudoconvex manifold (M^{2n+1}, H, J, θ) and (B,h) is locally isometric to the Fefferman space (F, h_θ) of (M, H, J, θ).

Proof. The proof runs through the same lines as in the *Spin*–case in [Bau99] and references given there: First, we prove that the Schouten tensor $K := K^h$ of (B,h) satisfies

$$K(V,V) = \text{const.} < 0. \tag{7.24}$$

To this end, we calculate using (7.10)

$$V \cdot \nabla^A_V D^A \varphi = \frac{n}{2} V \cdot K(V) + c_1 \cdot V \cdot (V \lrcorner dA) \cdot \varphi + c_2 \cdot V \cdot (V^\flat \wedge dA) \cdot \varphi,$$

where the real constants $c_{1,2}$ are specified by (7.10). However, as V is lightlike and $V \lrcorner dA = 0$, the last two summands vanish by (7.15). Consequently,

$$V \cdot \nabla^A_V D^A \varphi = \frac{n}{2} V \cdot K^g(V) \stackrel{1.}{=} -n \cdot K(V,V) \cdot \varphi.$$

On the other hand, the twistor equation and our assumptions yield

$$V \cdot \nabla^A_V D^A \stackrel{1.}{=} \nabla^A_V (V \cdot D^A) = -n \cdot \nabla^A_V \nabla^A_V \varphi \stackrel{4.}{=} nc^2 \cdot \varphi.$$

Consequently, $K(V,V) = -c^2$ and (7.24) holds.
Regularity of V implies that there is a natural S^1-action on B,

$$B \times S^1 \ni (p, e^{it}) \mapsto \gamma^V_{t \cdot \frac{L}{2\pi}}(p) \in B,$$

where $\gamma^V_t(p)$ is the integral curve of V through p and L is the period of the integral curves. Thus, $M := B/S^1$ is a $2n+1$-dimensional manifold and V is the fundamental vector field defined by the element $\frac{2\pi}{L} i \in i\mathbb{R}$ in the S^1-principal bundle $(B, \pi, M; S^1)$.
As V is by assumption normal and satisfies (7.24), Sparlings characterization of Fefferman spaces applies (see [Gra87]), yielding that there is a strictly pseudoconvex pseudo-Hermitian structure (H, J, θ) on M such that (B,h) is locally isometric to the Fefferman space (F, h_θ) of (M, H, J, θ). For more details regarding the construction of the local isometries $\phi_U : B_{|U} \to F_{|U}$ we refer to [Bau99, Gra87]. □

Remark 7.26 The last two statements yield a $Spin^c$–twistor spinor characterization of Fefferman spaces. It seems very natural to characterize Fefferman spaces in terms of distinguished $Spin^c$-spinor fields as every Fefferman space over a strictly pseudoconvex CR-manifold admits a natural $Spin^c$-structure. For a characterization in terms of ordinary twistor spinors, one has to restrict to the class of Fefferman *spin*-spaces, see [Bau99].

Finally, we remark that the results of this section help to complete the following diagram of geometric structures admitting special spinor fields:

[7] From this condition, it follows with Lemma 1.24 that $V \cdot \varphi = 0$.

$\boxed{(M^{2n}, g) \text{ Ricci-flat Kähler}}$ 2 *parallel spinors*, $Hol(M,g) \subset SU(n)$	$\xrightarrow{\text{conformal analogue}}$	$\boxed{(F^{2n+2}, h_\theta) \text{ Feff. spin space}}$ 2 *Twistor spinors*, $Hol(F,[h_\theta]) \subset SU(1,n)$
$\downarrow$ *Spin*^{c}-analogue		$\downarrow$ *Spin*^{c}-analogue
$\boxed{(M^{2n}, g) \text{ Kähler}}$ 1 *Spin*^{c} – *parallel spinor*, $Hol(M,g) \subset U(n)$, $dA = i \cdot Ric(\cdot, J\cdot)$	$\xrightarrow{\text{conformal analogue}}$	$\boxed{(F^{2n+2}, h_\theta) \text{ Feff. space}}$ 1 *CCKS*, $Hol(F,[h_\theta]) \subset SU(1,n)$, $dA = \pi^* Ric^W$

7.5 A partial classification result for the Lorentzian case

We present a complete description of Lorentzian manifolds admitting a CCKS under the additional assumption that V_φ is *normal*. In the *Spin*-case this is always satisfied. The proof of the next statement closely follows the *Spin*–case from [Lei07]. Recall that for a 1-form $\alpha \in \Omega^1(M)$ we define the rank of α to be $rk(\alpha) := \max\{n \in \mathbb{N}_0 \mid \alpha \wedge (d\alpha)^n \neq 0\}$.

Theorem 7.27 *Let (M,g) be a Lorentzian Spin^c-manifold admitting a nontrivial CCKS $\varphi \in \Gamma(S^g)$ wrt. a S^1–connection $A \in \Omega^1(\mathcal{P}_1, i\mathbb{R})$. Assume further that $V := V_\varphi \in \mathfrak{X}(M)$ is a normal conformal vector field. Then locally off a singular set exactly one of the following cases occurs:*

1. *It holds that $rk(V^\flat) = 0$ and $\|V\|_g^2 = 0$.*
   The spinor φ is locally conformally equivalent to a Spin^c-parallel spinor on a Brinkmann space.
2. *It holds that $rk(V^\flat) = 0$ and $\|V\|_g^2 < 0$.*
   Locally, $[g] = [-dt^2 + h]$, where h is a Riemannian metric admitting a Spin^c-parallel spinor. The latter metrics are completely classified, cf. [Mor97].
3. *n is odd and $rk(V^\flat) = (n-1)/2$ is maximal.*
   (M,g) is locally conformally equivalent to a Lorentzian Einstein Sasaki manifold[8]. There exist geometric Spin-Killing spinors $\varphi_{1,2}$ on (M,g) which might be different from φ, but satisfying $V_{\varphi_{1,2}} = V$.
4. *n is even and $rk(V^\flat) = (n-2)/2$ is maximal.*
   In this case, (M,g) is locally conformally equivalent to a Fefferman space.
5. *If none of these cases occurs, there exists locally a product metric $g_1 \times g_2 \in [g]$, where g_1 is a Lorentzian Einstein Sasaki metric on a space M_1 admitting a geometric Killing spinor φ_1 and g_2 is a Riemannian Einstein metric on a space M_2 such that $M = M_1 \times M_2$ and $V = V_{\varphi_1}$.*

Conversely, given one of the above geometries with a CCKS of the mentioned type, the associated Dirac current V is always normal.

[8]Note that every simply-connected Einstein Sasaki manifold is spin, see [Boh99].

Proof. The condition that V is normal is equivalent to say that α^1_φ is a normal conformal Killing 1-form (cf. Remark 7.14), which means that the RHS in (7.19) vanishes. As elaborated on in section 2.4, this is equivalent to have a parallel tractor 2-form, i.e. there exists a 2-form $\alpha \in \Lambda^2_{2,n}$ which is fixed by the conformal holonomy representation $Hol(M,c) \subset SO^+(2,n)$. The system of equations (7.19) allows us to conclude as in section 3.1 that $\alpha = \alpha^2_\chi$ for a spinor $\chi \in \Delta_{2,n}$[9]. 2-forms induced by a spinor in signature $(2,n)$ have been classified in [Lei07], we have cited the result in Remark 1.25, and the geometric meaning of a holonomy-reduction imposed by such a fixed α^2_χ is well-understood. The following possibilities can occur:

$\alpha = l_1^\flat \wedge l_2^\flat$ for l_1, l_2 mutually orthogonal lightlike vectors. It follows from Lemma (3.8) there is locally a metric $g \in c$ on $U \subset M$ such that V is lightlike and parallel. It remains to prove that also φ is ∇^A-parallel in this situation: It follows from Lemma 1.24 that $V \cdot \varphi = 0$. Differentiating yields that $V \cdot \nabla^A_X \varphi = 0$. Let $(s_1, ..., s_n)$ be a local pseudo-orthonormal frame with $V = s_1 + s_2$. Then it follows by Clifford multiplication with s_1 that

$$s_1 \cdot s_2 \cdot \nabla^A_X \varphi = -\nabla^A_X \varphi \text{ for all } X \in TU.$$

As φ is a twistor spinor, the spinor $\phi := g(X,X) X \cdot \nabla^A_X \varphi$ does not depend on the choice of the vector field X with $g(X,X) = \pm 1$. Let$X \in V^\perp$ with $g(X,X) = \pm 1$. It follows that $0 = -2g(X,V) = X \cdot V + V \cdot X$, and therefore, $V \cdot \phi = -g(X,X) X \cdot V \cdot \nabla^A_X \varphi = 0$ for all $X \in V^\perp$. On the other hand, choosing $X = s_1$ yields

$$0 = V \cdot \phi = -(s_1 + s_2) \cdot s_1 \cdot \nabla^A_{s_1} \varphi = (-1 + s_1 \cdot s_2) \cdot \nabla^A_{s_1} \varphi = -2 \nabla^A_{s_1} \varphi.$$

Consequently, $\phi = 0$, and therefore also $\nabla^A \varphi = 0$.

$\alpha = l^\flat \wedge t^\flat$, where l is a lightlike vector and t a orthogonal timelike vector. [Lei07] shows that there is locally a Ricci-flat metric in the conformal class on which V is parallel and timelike. By constantly rescaling the metric, we may assume that $||V||^2 = -1$. We have to show that the spinor itself is parallel in this situation. To this end, we calculate:

$$\begin{aligned} 0 = Vg(V,V) = V\langle V \cdot \varphi, \varphi \rangle &= -\frac{1}{n}(\langle V^2 \cdot D^g \varphi, \varphi \rangle + \langle V \cdot \varphi, V \cdot D^g \varphi \rangle) \\ &= -\frac{2}{n} \mathrm{Re} \langle D^g \varphi, \varphi \rangle \end{aligned}$$

We differentiate this function wrt. an arbitrary vector X, use $K^g = 0$ and (7.10) to obtain

$$0 = \mathrm{Re}\langle c_1 (X \lrcorner dA) \cdot \varphi + c_2 (X^\flat \wedge dA) \cdot \varphi, \varphi \rangle - \frac{1}{n} \mathrm{Re} \langle X \cdot D^A \varphi, D^A \varphi \rangle.$$

The first scalar product vanishes as $\langle (X \lrcorner dA) \cdot \varphi, \varphi \rangle \in i\mathbb{R}$ and $\langle (X^\flat \wedge dA) \cdot \varphi, \varphi \rangle = 0$ by Proposition 7.17. Thus, $0 = V_{D^A \varphi}$ from which in the Lorentzian case $D^A \varphi = 0$ follows. It is clear that φ descends to a $Spin^c$–parallel spinor on the Riemannian factor.

n is odd and $\alpha = (\omega_0)_{|V}$, where $V \subset \mathbb{R}^{2,n}$ is a pseudo-Euclidean subspace of signature $(2, n-1)$ and ω_0 denotes the pseudo-Kähler form on V. In this case $Hol(M,c) \subset SU(1, (n-1)/2)$. [Lei07] shows in Theorem 10 that there is locally a Lorentzian Einstein

[9]This argument can also be found in [Lei09].

Sasaki metric g (of negative scalar curvature) in the conformal class. Moreover, V is unit timelike Killing wrt. this metric and belongs to the defining data of the Sasakian structure. It is known from [Boh99] that there are geometric Killing spinors φ_i on (M, g) with $V_{\varphi_i} = V$.

n is even and $\alpha = \omega_0$ is the pseudo-Kähler form on $\mathbb{R}^{2,n}$. This corresponds to conformal holonomy in $SU(1, n/2)$ and as known from [Lei07] this is locally equivalent to having a Fefferman space in the conformal class on which a CCKS exists by the preceding section.

$\alpha = (\omega_0)_{|W}$, where $W \subset \mathbb{R}^{2,n}$ is a pseudo-Euclidean subspace of even dimension and signature $(2, k)$, where $4 \leq k < n-2$ and ω_0 denotes the pseudo-Kähler form on W. In this case, the conformal holonomy representation fixes a proper, nondegenerate subspace of dimension ≥ 2 and is special unitary on the orthogonal complement. As shown in [Lei07] this is exactly the case if locally there is a metric in the conformal class such that $(M, g) = (M_1 \times M_2, g_1 \times g_2)$, where the first factor is Lorentzian Einstein Sasaki. As mentioned before, there exists a geometric *Spin*–Killing spinor inducing V on M_1.

Conversely, if one of the geometries from Theorem 7.27 together with a $Spin^c$–CCKS of mentioned type as in the Theorem is given, it follows that V_φ is normal conformal: In the first two cases, φ is parallel, for which $Ric(X) \cdot \varphi = 1/2(X \lrcorner dA) \cdot \varphi$ is known (see [Mor97]). We thus have that $(X \lrcorner dA)^\sharp \lrcorner \alpha^3_\varphi \in i\mathbb{R} \cap \mathbb{R} = \{0\}$. Proposition 7.17 yields that $V_\varphi \lrcorner W^g = 0$. A analogous straightforward but tedious equation yields that $V_\varphi \lrcorner C^g = 0$. In cases 3 and 5 of Theorem 7.27, V is normal as it is the Dirac current of a *Spin*–Killing spinor. Case 4 was discussed in the previous section and V is normal by Theorem 7.23. □

Remark 7.28 We remark that the *Spin*-Killing spinors φ_i in cases 3 and 5 might be different from the spinor φ we started with, i.e. it could be the case that on the Lorentzian Einstein Sasaki space, the original spinor φ is a CCKS wrt. some nontrivial connection A. However, as shown in [Mor97], if (M, g) is an irreducible Lorentzian Einstein Sasaki manifold, only $Spin^c$ structures with $dA = 0$ admit Killing spinors.

Remark 7.29 The classification for the Riemannian case seems to differ drastically from the *Spin*–case. For instance, a CCKS on a Ricci-flat manifold need not be parallel and the CCKS equation does not reduce to the study of parallel or Killing spinors on conformally related metrics as in the spin case. Furthermore, every Riemannian 3-manifold admitting a twistor spinor is conformally flat (see [BFGK91]), whereas there are examples of 3-dimensional non-conformally flat $Spin^c$-manifolds admitting CCKS which can not be rescaled to parallel or Killing spinors, see [GN13].

7.6 Low dimensions

A geometric motivation

In physics literature, conformal structures admitting CCKS have been classified for Riemannian and Lorentzian manifolds of dimensions 3 and 4, see [CKM+14, HTZ13, KTZ12, CM13]. Interestingly, one observes that CCKS yield a *spinorial characterization for the existence of certain conformal tensors* in these signatures. Let us motivate the classification of low dimensional conformal structures admitting a CCKS from this geometric point of view, taking signature $(3,1)$ as an example. To this end, consider the map

$$l : \Delta_{3,1}^{\mathbb{C},+}\backslash\{0\} \cong \mathbb{C}^2\backslash\{0\} \to L^+ \subset \mathbb{R}^{3,1},\ \epsilon \mapsto V_\epsilon,$$

where L^+ denotes the forward lightcone. This map is surjective (cf. [Lei01]) and the space $\{\epsilon \in \Delta_{3,1}^{\mathbb{C},+} \mid (\epsilon,\epsilon)_{\mathbb{C}^2} = \text{const.} > 0\}$ is an S^3 which is mapped by l to the space of null vectors $z \in \mathbb{R}^{3,1}$ with fixed space component z_4, i.e. the image is an S^2. Thus, l restricts to the Hopf fibration map with fibre $S^1 \cong U(1)$. Similarly one can show that $\Delta_{3,1}^{\mathbb{R}}\backslash\{0\}/S^1 \cong L^+$. In the spin case, one uses this last observation to prove:

Theorem 7.30 ([Lei01]) *Let $(M^{3,1},g)$ be a non-conformally flat Lorentzian manifold admitting a null normal conformal vector field V without zeroes such that its twist $V^\flat \wedge dV^\flat$ vanishes everywhere or nowhere on M. Then there exists locally a real twistor spinor $\varphi \in \Gamma(S^g_{\mathbb{R}})$ such that $V_\varphi = V$.*

Thus, in signature $(3,1)$ real twistor spinors locally characterize the existence of normal conformal null vector fields with a certain twist condition. In view of this, we ask whether the existence of a generic null conformal vector field on $(M^{3,1},g)$ which is not necessarily normal conformal can be characterized in terms of spinor fields. As passing from a null vector field V to a *complex* half spinor field $\varphi \in \Gamma(S^g_{\mathbb{C}})$ via the map l comes with a $U(1)$-ambiguity at each point, i.e. $V = V_\varphi$ iff $V = V_{f\varphi}$ for every $f : M \to S^1$, it seems natural to include a S^1−*gauge field* A which precisely gauges this symmetry. By (7.5), this leads to *Spin*^{c}-geometry. Indeed, one can now prove the following:

Proposition 7.31 ([CKM+14]) *Let V be a null conformal vector field without zeroes on a Lorentzian manifold $(M^{3,1},g)$. Then there exists locally a connection A and a CCKS $\varphi \in \Gamma(S^g_{\mathbb{C},+})$ wrt. A such that $V = V_\varphi$.*

Likewise, on a 3-dimensional Lorentzian manifold the existence of a CCKS without zeroes is locally equivalent to the existence of a null or timelike conformal vector field (cf. [HTZ13]). Also in Riemannian signature $(4,0)$ and $(3,0)$ the existence of a CCKS yields an equivalent spinorial characterization of natural geometric structures, see [KTZ12]. The signatures $(2,1)$ and $(3,1)$ in mind, we hope that also in higher (Lorentzian) signatures CCKS might locally characterize the existence of certain conformal, but not necessarily normal conformal tensors. This is indeed the case as we shall see in the next sections.

Remark 7.32 In the following, all of our considerations will be *local* on some open, simply-connected set $U \subset M$, i.e. we can always assume that there is a uniquely determined *Spin*−structure, the S^1-bundle is trivial and under this identification A corresponds to a 1-form $A \in \Omega^1(U, i\mathbb{R})$.

5-dimensional Lorentzian manifolds with a CCKS

The spinor representation in signature $(1,4)$ is quaternionic, i.e. $\Delta_{1,4} := \Delta^{\mathbb{C}}_{1,4} \cong \mathbb{H}^2$. However, we prefer to work with complex quantities. A Clifford representation on $\mathbb{C}^4$ is given by:

$$e_1 = \begin{pmatrix} & i & & \\ i & & & \\ & & & i \\ & & i & \end{pmatrix}, e_2 = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, e_3 = \begin{pmatrix} & & -i & \\ & & & i \\ -i & & & \\ & i & & \end{pmatrix}, e_4 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix},$$
$$e_0 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \tag{7.25}$$

The $Spin^c(1,4)$-invariant scalar product is given by $\langle v, w\rangle_{\Delta_{1,4}} = (e_0 \cdot v, w)_{\mathbb{C}^4}$. According to [Bry00], the nonzero orbits of the action of $Spin^+(1,4) \cong Sp(1,1)$ on $\Delta_{1,4}$ are just the level sets of $v \mapsto \langle v, v\rangle_{\Delta_{1,4}} \in \mathbb{R}$. Consider the spinor $u_1 = \begin{pmatrix} 1 & 0 & 0 & 0\end{pmatrix}^T$. It satisfies

$$\langle u_1, u_1\rangle_{\Delta_{1,4}} = 1,\ V_{u_1} = e_0,\ \alpha^2_{u_1} = e_1^\flat \wedge e_2^\flat + e_3^\flat \wedge e_4^\flat,\ \alpha^2_{u_1} \cdot u_1 = 2i \cdot u_1, \tag{7.26}$$

whereas for the spinor $u_0 = \begin{pmatrix} 1 & 1 & 0 & 0\end{pmatrix}^T \in \Delta_{1,4}$ we find

$$\langle u_0, u_0\rangle_{\Delta_{1,4}} = 0,\ V_{u_0} = -2(e_0 + e_2),\ \alpha^2_{u_0} = 2(e_1^\flat \wedge (e_0^\flat + e_2^\flat)),\ \alpha^2_{u_0} \cdot u_0 = 0. \tag{7.27}$$

Here, $\langle \alpha^2_u, \alpha\rangle_{1,4} = \frac{1}{i}\langle \alpha \cdot u, u\rangle_{\Delta_{1,4}} \in \mathbb{R}$ for $\alpha \in \Lambda^2_{1,4}$.

Let $(M^{1,4}, g)$ be a Lorentzian $Spin^c$-manifold admitting a CCKS φ wrt. a S^1–connection A. Locally, around a given point, one has by omitting singular points either that $\langle \varphi, \varphi\rangle \neq 0$ or $\langle \varphi, \varphi\rangle \equiv 0$. In the first case let us assume that $\langle \varphi, \varphi\rangle > 0$. The analysis for CCKS of negative length is completely analogous. Thus, locally there are only two cases to consider:

Case 1:
We may after rescaling the metric assume that $\varphi \in \Gamma(S^g)$ is a CCKS with $\langle \varphi, \varphi\rangle \equiv 1$. Differentiating the length function and inserting the twistor equation yields that

$$\mathrm{Re}\langle X \cdot \varphi, \eta\rangle \equiv 0, \tag{7.28}$$

where $\eta := -\frac{1}{5}D^A\varphi$. Let $s = (s_0, ..., s_4)$ be a local orthonormal frame with lift $\widetilde{s}$ to the spin structure (cf. Remark 7.32) such that locally $\varphi = [\widetilde{s}, u_1]$, $V_\varphi = [s, e_0]$, $\alpha^2_\varphi = [s, \alpha^2_{u_1}]$ and $\eta = \left[\widetilde{s}, \begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \\ a_5 + ia_6 \\ a_7 + ia_8 \end{pmatrix}\right]$ for functions $a_1, ..., a_8 : U \to \mathbb{R}$. However, (7.28) is satisfied iff

$$\eta = \left[\widetilde{s}, \begin{pmatrix} ia_2 & 0 & 0 & a_7 + ia_8\end{pmatrix}^T\right].$$

With this preparation, the conformal Killing equation for α^2_φ (cf. the first line of (7.19)) is straightforwardly calculated to be

$$\nabla^g_X \alpha^2_\varphi = \text{const.} \cdot \mathrm{Im}\ \langle \varphi, D^g\varphi\rangle_{S^g} \cdot X^\flat \wedge V^\flat_\varphi.$$

In particular, $d\alpha^2_\varphi = 0$ and $\nabla^g_{V_\varphi}\alpha^2_\varphi = 0$ in this scale. We now differentiate $\alpha^2_\varphi \cdot \varphi = 2i \cdot \varphi$ wrt. V_φ to obtain $\alpha^2_\varphi \cdot \nabla^A_{V_\varphi}\varphi = 2i\nabla^A_{V_\varphi}\varphi$. We multiply this equation by V_φ. By (7.26) the actions of α^2_φ and V_φ on spinors commute. Furthermore, $V_\varphi \cdot \nabla^A_{V_\varphi}\varphi = \eta$ by the twistor equation, leading to $\alpha^2_\varphi \cdot \eta = 2i\eta$, i.e.

$$\alpha^2_\varphi \cdot \eta = 2i\left[\widetilde{s}, \begin{pmatrix} ia_2 & 0 & 0 & -a_7 - ia_8 \end{pmatrix}^T\right] = 2i\eta = 2i\left[\widetilde{s}, \begin{pmatrix} ia_2 & 0 & 0 & a_7 + ia_8 \end{pmatrix}^T\right].$$

Consequently, $D^A\varphi = -5ia_2 \cdot \varphi$, and thus $\nabla^A_X\varphi = ia_2 \cdot X \cdot \varphi$. However, it is proven in [HM99] that this forces a_2 to be constant, i.e. φ is a *Spin*^{c}-Killing spinor or a *Spin*^{c}-parallel spinor. In the second case, V_φ is parallel as well and the metric splits into a product $(\mathbb{R}, -dt^2) \times (N, h)$ where the Riemannian 4-manifold (N,h) admits a parallel *Spin*^{c}-spinor. As moreover α^2_φ descends to a parallel 2-form on (N,h) of Kähler type, we conclude that (N,h) is Kähler. Conversely, every Kähler *Spin*^{c}-manifold endowed with its canonical *Spin*^{c}-structure admits parallel spinors. If φ is an imaginary Killing spinor, $\mathrm{Re}\langle\varphi, D^A\varphi\rangle = 0$, thus V_φ is a timelike Killing vector field of unit length satisfying $V_\varphi \cdot \varphi = \varphi$. By a constant rescaling of the metric we may moreover assume that the Killing constant is given by $\pm\frac{i}{2}$. Then it is known from [Boh99], Thm. 46, that V_φ defines a (not necessarily Einstein) Lorentzian Sasaki structure. Conversely, by [Mor97] every Lorentzian Sasaki structure endowed with its canonical *Spin*^{c}-structure admits imaginary *Spin*^{c}-Killing spinors.

Case 2:
Let us turn to the case in which the CCKS satisfies $\langle\varphi,\varphi\rangle \equiv 0$. We first remark that in the *Spin*-case, i.e. $A \equiv 0$, this always implies that the spinor is locally conformally equivalent to a parallel spinor off a singular set (see [Lei01], Lemma 4.4.6). As we shall see, in the *Spin*^{c}-case something more interesting happens: By passing to a dense subset we may assume that φ and V_φ have no zeroes. We locally rescale the metric such that V_φ becomes Killing[10] which is by (7.18) equivalent to

$$\mathrm{Re}\langle\varphi, D^A\varphi\rangle = 0 \tag{7.29}$$

in this metric g. In the chosen metric we also have (see (7.27)) that $\alpha^2_\varphi = r^\flat \wedge V^\flat_\varphi$, where r is a spacelike vector field of constant length orthogonal to V_φ. Proceeding exactly as in the first case, i.e. locally evaluating the conditions $\mathrm{Re}\langle X \cdot \varphi, D^A\varphi\rangle \equiv 0$ (resulting from differentiating the length function) and (7.29) and inserting this into the definition (7.17) for $\alpha^1_\mp$ and using the conformal Killing equation for α^2_φ leads to

$$\alpha^1_\mp = \mathrm{const.}_1 \cdot d^*\alpha^2_\varphi = \mathrm{const.}_2 \cdot \mathrm{Im}\langle\varphi, D^A\varphi\rangle_{S^g} \cdot V^\flat_\varphi, \tag{7.30}$$

for some real nonzero constants. Conversely, a local computation shows that given a conformal Killing form $\alpha = r^\flat \wedge l^\flat$ such that $\alpha_\mp = f \cdot l^\flat$ and r is spacelike, orthogonal to l and of constant length, then l has to be a Killing vector field. We summarize:

Proposition 7.33 *Given a CCKS $\varphi \in \ker P^A$ without zeroes such that $\langle\varphi,\varphi\rangle \equiv 0$, the conformal Killing form α^2_φ satisfies $\alpha^2_\varphi = r^\flat \wedge V^\flat_\varphi$ for a spacelike vector field r. There is a local metric $g \in c$ such that $d^*\alpha^2_\varphi = const. \cdot Im\langle\varphi, D^A\varphi\rangle_{S^g} \cdot V^\flat_\varphi$. In this scale, V_φ is Killing.*

[10] Choose local coordinates such that $V = \partial_1$. If g is any metric in the conformal class, we have that $L_V g = \lambda g$ for a function λ. V being Killing wrt. $e^{2\sigma}g$ is equivalent to $\partial_1\sigma = -\frac{\lambda}{2}$ which can be solved locally for σ.

We will now prove that the converse is also true, i.e. we show:

Proposition 7.34 *Given a zero-free conformal Killing 2-form $\alpha = r^\flat \wedge l^\flat \in \Omega^2(M)$ where $r \in \mathfrak{X}(M)$ is spacelike and of unit length, $l \in \mathfrak{X}(M)$ is a orthogonal lightlike vector field such that $d^*\alpha = f \cdot l^\flat$, for some function f, then there exists locally $A \in \Omega^1(U, i\mathbb{R})$ and a CCKS $\varphi \in \Gamma(U, S^g_{\mathbb{C}})$ wrt. A such that $\alpha^2_\varphi = \alpha$ and $f = const. \cdot Im\langle \varphi, D^A\varphi\rangle_{S^g}$.*

Proof. There exists a local orthonormal frame $s = (s_0, ..., s_4)$ such that locally $\alpha = [s, \alpha^2_{u_0}]$. Defining $\varphi = [\widehat{s}, u_0]$, where $\widehat{s}$ is the local lift of s to the spin structure shows that $\alpha^2_\varphi = \alpha$ and $\alpha = r^\flat \wedge V^\flat_\varphi$. It is a purely algebraic observation that φ is the up to local $U(1)$-action unique spinor field with this property, i.e. the surjective map

$$\Delta^{\mathbb{C}}_{1,4} \supset \{\epsilon \mid \langle \epsilon, \epsilon\rangle_\Delta = 0\} \mapsto \alpha^2_\epsilon \in \{\alpha \mid \alpha = r^\flat \wedge l^\flat, ||r||^2_{1,4} = 1, ||l||^2_{1,4} = 0, \langle r, l\rangle_{1,4} = 0\} \subset \Lambda^2_{1,4}$$

is an S^1-fibration. Locally, the mentioned properties of α give a linear system of equations for the local connection coefficients ω_{ij}. By the local formula (7.4) the property of φ being a CCKS becomes a linear system of equations for the ω_{ij} and the $A_i := A(s_i) \in C^\infty(U, i\mathbb{R})$. A tedious but straightforward computation shows that there is a unique choice of A such that these equations are indeed satisfied. In our chosen gauge one has that

$$\begin{aligned} &A_1 = -2i\omega_{34}(s_1), A_2 = -2i\omega_{34}(s_2), A_3 = -2i(\omega_{34}(s_3) + \omega_{14}(s_3)), \\ &A_4 = -2i(\omega_{34}(s_4) + \omega_{14}(s_4)), A_0 = -2i\omega_{34}(s_0). \end{aligned} \tag{7.31}$$

In detail, this argument goes as follows: We have to show that if the locally given 2- form $\alpha = \alpha^2_\varphi = [s, \alpha^2_{u_0}] = s^\flat_1 \wedge (s^\flat_2 + s^\flat_0)$ where $s = (s_0, ..., s_4)$ is a local ONB, is a conformal Killing 2-form such that $\alpha_\mp = \widetilde{f} \cdot V^\flat_\varphi = \widetilde{f} \cdot [s, e^\flat_2 + e^\flat_0] = \widetilde{f} \cdot (s^\flat_2 + s^\flat_0)$ for some function $\widetilde{f}$, then there is a uniquely determined $A \in \Omega^1(U, i\mathbb{R})$ such that the spinor $\varphi = [\widetilde{s}, u_0]$ is a CCKS wrt. A[11]. To this end, note that by the equivalent characterization of conformal Killing forms in [Sem01], the requirement on α is equivalent to

$$X \lrcorner \nabla^g_Y \alpha + Y \lrcorner \nabla^g_X \alpha = f \cdot \left(X \lrcorner (Y^\flat \wedge V^\flat_\varphi) + Y \lrcorner (X^\flat \wedge V^\flat_\varphi)\right) \ \forall X, Y \in TM, \tag{7.32}$$

where $f = \text{const.} \cdot \widetilde{f}$. We let X, Y run over the local ONB $(s_0, s_1, s_2, s_3, s_4)$ and use

$$\nabla^g_X (s^\flat_i \wedge s^\flat_k) = \sum_j \epsilon_i \omega_{ij}(X) s^\flat_j \wedge s^\flat_k + \sum_j \epsilon_k \omega_{kj}(X) s^\flat_i \wedge s^\flat_j$$

to obtain that (7.32) is the following system of linear equations in $\omega^k_{ij} := \epsilon_i \epsilon_j g(\nabla_{s_k} s_i, s_j)$:

[11]By (7.30) we then necessarily have that $\widetilde{f}$ is a constant multiple of Im $\langle D^A\varphi, \varphi\rangle_{S^g}$

$$
\begin{aligned}
&\omega^1_{20} = f, \omega^1_{23} + \omega^1_{30} = 0, \omega^1_{24} + \omega^1_{40} = 0,\\
&\omega^2_{20} = 0, \omega^1_{12} - \omega^1_{10} = 0, \omega^2_{24} + \omega^2_{40} = 0, \omega^2_{23} + \omega^2_{30} = \omega^1_{13},\\
&\omega^1_{13} = -\omega^3_{20}, \omega^3_{23} + \omega^3_{30} = 0, \omega^3_{24} + \omega^3_{40} = 0,\\
&\omega^1_{14} + \omega^4_{20} = 0, \omega^4_{23} + \omega^4_{30} = 0,\\
&\omega^0_{20} = \omega^1_{10} - \omega^1_{12}, \omega^0_{23} + \omega^0_{30} = -\omega^1_{13}, \omega^0_{24} + \omega^0_{40} = -\omega^1_{14}, \omega^0_{20} = 0,\\
&\omega^2_{13} = 0, \omega^2_{14} = 0, \omega^2_{12} - \omega^2_{10} = f,\\
&\omega^2_{23} + \omega^2_{30} = -\omega^3_{20}, \omega^3_{13} = f, \omega^3_{14} = 0, \omega^3_{12} - \omega^3_{10} = 0,\\
&\omega^2_{24} = -\omega^4_{20}, \omega^2_{14} = 0, \omega^4_{13} = 0, \omega^4_{14} = f, \omega^4_{12} - \omega^4_{10} = 0,\\
&\omega^2_{13} = \omega^0_{13}, \omega^0_{14} = \omega^2_{14}, \omega^0_{12} - \omega^0_{10} = -f,\\
&\omega^3_{23} + \omega^3_{30} = 0, \omega^3_{13} = f, \omega^3_{14} = -\omega^4_{13},\\
&\omega^3_{20} = \omega^0_{23} + \omega^0_{30}, \omega^0_{13} = \omega^3_{10} - \omega^3_{12}, \omega^3_{13} = f, \omega^3_{14} = 0, \omega^0_{13} = 0,\\
&\omega^4_{24} + \omega^4_{40} = 0, \omega^4_{14} = f,\\
&\omega^0_{24} + \omega^0_{40} = \omega^4_{20}, \omega^4_{12} - \omega^4_{10} = -\omega^0_{14}, \omega^0_{14} = 0,\\
&\omega^0_{12} - \omega^0_{10} = 0.
\end{aligned}
\tag{7.33}
$$

(7.33) exhibit that V_φ is a Killing vector field, being equivalent to the equations

$$\epsilon_j(\omega^i_{2j} - \omega^i_{0j}) + \epsilon_i(\omega^j_{2i} - \omega^j_{0i}) = 0.$$

On the other hand, by the local formula (7.4), the twistor equation for φ is equivalent to

$$\epsilon_i e_i \cdot \left(\sum_{k<l} \omega^i_{kl} e_k \cdot e_l \cdot u_0 + \frac{1}{2} A_i \cdot u_0 \right) = \epsilon_j e_j \cdot \left(\sum_{k<l} \omega^i_{kl} e_k \cdot e_l \cdot u_0 + \frac{1}{2} A_j \cdot u_0 \right), \tag{7.34}$$

for $0 \le i < j \le 4$ and $A_i := A(s_i) : U \to i\mathbb{R}$. Inserting the above ω–equations, it is pure linear algebra to check that (7.34) holds if and only if we set the local functions A_i as given in (7.31). □

We summarize our observations:

Theorem 7.35 *Let $\varphi \in \Gamma(M, S^g)$ be a CCKS wrt. a connection A on a Lorentzian 5-manifold (M, g). Locally and off a singular set the metric can be rescaled such that exactly one of the following cases occurs:*

1. *The spinor is of nonzero length and a parallel Spin^c-spinor on a metric product $\mathbb{R} \times N$, where N is a Riemannian 4-Kähler manifold with parallel spinor.*
2. *φ is an imaginary Spin^c-Killing spinor of nonzero length, its vector field V_φ is Killing and defines a Sasakian structure.*
3. *$|\varphi|^2 \equiv 0$. The conformal Killing form $\alpha^2_\varphi =: \alpha$ can be written as $\alpha = r^\flat \wedge l^\flat$, where $r \in \mathfrak{X}(M)$ is spacelike and $l \in \mathfrak{X}(M)$ is orthogonal to r and lightlike. There is a scale in which $d^*\alpha = f \cdot l^\flat$ for some function f.*

Conversely, for all the geometries listed in 1.-3. there exists (in case 3. only locally) a Spin^c-structure, a S^1–connection A and a CCKS $\varphi \in ker\ P^A$.

Remark 7.36 It is easy to verify that the correspondence in the third part of this Theorem descends to parallel objects, i.e. on a Lorentzian $Spin^c$-manifold $(M^{1,4}, g)$ there exists a $Spin^c$-parallel spinor of zero length if and only if there is a parallel 2-form of type $\alpha = l^\flat \wedge r^\flat$. This can be understood well from a holonomy-point of view: The $Spin^+(1,4)$-stabilizer of an isotropic spinor in signature $(1,4)$ is by [Bry00] isomorphic to $\mathbb{R}^3$, thus for a ∇^A-parallel spinor of zero length, we have $Hol(\nabla^A) \subset S^1 \cdot \mathbb{R}^3 \cong SO(2) \ltimes \mathbb{R}^3 \subset SO^+(1,4)$ which is precisely the stabilizer of a 2-form $\alpha = r^\flat \wedge l^\flat$ of causal type as above under the $SO^+(1,4)$–action. That means, $Hol(\nabla^A)$ fixes an isotropic spinor iff $Hol(\nabla^g)$ fixes a 2-form $\alpha = r^\flat \wedge l^\flat$.

Finally, the third case from Theorem 7.35 can with (7.30) be specialized and yields the following spinorial characterization of geometries admitting certain Killing forms, i.e. conformal Killing forms α with $d^*\alpha = 0$:

Theorem 7.37 *On every Lorentzian 5-manifold admitting a Killing 2-form of type $r^\flat \wedge l^\flat$ for a spacelike vector field r of unit length and a orthogonal lightlike vector field l, there exists (locally) a CCKS with $\langle \varphi, D^A\varphi \rangle_{S^g} = 0$ and vice versa.*

Other signatures

We investigate the CCKS-equation on manifolds of signature $(0,5), (2,2)$ and $(3,2)$. Together with the last section and the results from [CKM⁺14, HTZ13, KTZ12] this yields a complete local description of geometries admitting CCKS in all signatures for dimension ≤ 5.

Signature (0,5)
Let us start with the Riemannian 5-case. A Clifford representation of $Cl_{0,5}$ on $\Delta_{0,5} = \mathbb{C}^4$ is given by (7.25) where one has to replace the e_0–matrix by $-i \cdot e_0$ (see [BFGK91]). The $Spin^+(0,5) \cong Sp(2)$-invariant scalar product on $\Delta_{0,5}^{\mathbb{C}}$ is just the usual Hermitian product on $\mathbb{C}^4$ and the nonzero orbits of the $Spin^+(0,5)$-action on spinors are given by its level sets. Let us consider the spinor $u := \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$. We have that $V_u = e_0$, α_u^2 is the Kähler form on $\text{span}\{e_1, ..., e_4\}$ and $\alpha_u^2 \cdot u = 2i \cdot u$ Now exactly the same considerations as carried out for spinors of nonzero length in the Lorentzian case in the previous section reveal the following:

Theorem 7.38 *Let $\varphi \in \Gamma(S^g)$ be a CCKS of constant length on a 5-dimensional Riemannian $Spin^c$-manifold (M, g). Locally, exactly one of the following cases occurs:*

1. *There is a metric split of (M, g) into a line and a 4-dimensional Kähler manifold on which φ is parallel.*
2. *After a rescaling of the metric, φ is a $Spin^c$-Killing spinor to Killing number $\pm\frac{1}{2}$. V_φ is a unit-norm Killing vector field which defines a Sasakian structure.*

Conversely, these geometries, equipped with their canonical $Spin^c$ structures, admit $Spin^c$-parallel/Killing spinors.

Consequently, CCKS in signature $(0,5)$ locally equivalently characterize the existence of Sasakian structures or splits into a line and a Kähler 4-manifold in the conformal class.

Signature (2,2)

$Cl_{2,2} \cong \mathfrak{gl}(4,\mathbb{R})$, and thus the complex representation of $Cl^{\mathbb{C}}_{2,2}$ on $\Delta^{\mathbb{C}}_{2,2} = \mathbb{C}^4$ arises as a complexification of the real representation

$$e_1 = -\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, e_2 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{pmatrix}, e_3 = \begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, e_4 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

of $Cl_{2,2}$ on $\Delta^{\mathbb{R}}_{2,2} = \mathbb{R}^4$. In this realisation, $Spin^+(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and the indefinite scalar product on $\Delta^{\mathbb{C}}_{2,2}$ given by $(e_1 \cdot e_2 \cdot v, w)_{\mathbb{C}^4}$ satisfies $\langle v, v \rangle_\Delta \in i\mathbb{R}$. The nonzero orbits of the $Spin^c(2,2)$-action on $\Delta^{\mathbb{C},\pm}_{2,2}$ are given by the level sets of $\langle \cdot, \cdot \rangle_\Delta$ where half spinors of zero length are precisely the real half spinors $\Delta^{\mathbb{R},\pm}_{2,2}$, multiplied by elements of $S^1 \subset \mathbb{C}$. These algebraic observations lead to the following local analysis:

Let $(M^{2,2}, g)$ be a $Spin^c(2,2)$-manifold admitting a nontrivial CCKS halfspinor $\varphi \in \Gamma(S^g_{\mathbb{C},\pm})$ wrt. the S^1–connection A. As we are only interested in local considerations, we may (after passing to open neighbourhoods of a given point and omitting a singular set) assume that $||\varphi||^2 \equiv 0$ or $||\varphi||^2 \neq 0$ everywhere. In the first case, the $Spin^c(2,2)$-orbit structure shows that φ can be chosen to be a local section of $S^g_{\mathbb{R},\pm}$ (see also Proposition 7.5), i.e. there exists locally a pseudo-orthonormal frame $s = (s_1, ...s_4)$ with lift $\widetilde{s}$ such that $\varphi = [\widetilde{s}, u_{0,\pm}]$ for some fixed spinor $u_{0,\pm} \in \Delta^{\mathbb{R},\pm}_{2,2}$. As φ is a CCKS, we must have that

$$\epsilon_i s_i \cdot \nabla^A_{s_i} \varphi = \epsilon_j s_j \cdot \nabla^A_{s_j} \varphi \in \Gamma(S^g_{\mathbb{C},\pm} = S^g_{\mathbb{R},\pm} \oplus i S^g_{\mathbb{R},\pm}) \ \forall 1 \leq i, j \leq 4. \tag{7.35}$$

Using the local formula (7.4) and splitting (7.35) into real and imaginary part, we arrive at $\epsilon_i A(s_i) \cdot s_i = \epsilon_j A(s_j) \cdot s_j$ which is possible only if $A \equiv 0$. Consequently, we are dealing with real $Spin^+(2,2)$ twistor half spinors which have been shown to be locally conformally equivalent to parallel spinors in chapter 5.

If, on the other hand, the spinor norm is nonvanishing, we may rescale the metric such that $||\varphi||^2 = \pm i$. Differentiating yields that Im $\langle X \cdot \varphi, D^A \varphi \rangle_{S^g} \equiv 0$ for $X \in TM$. It is purely algebraic to check that this is possible only if $D^A \varphi = 0$. Moreover, α^2_φ is a constant multiple of the pseudo-Kähler form, i.e. φ is a $Spin^c$-parallel half spinor on a Kähler manifold of signature $(2,2)$. We summarize:

Theorem 7.39 *Let $\varphi \in \Gamma(S^g_{\mathbb{C},\pm})$ be a CCKS on a $Spin^c$-manifold $(M^{2,2}, g)$. Locally, one of the following cases occurs:*

1. *$||\varphi||^2 = 0$. This implies $A \equiv 0$. The spinor can be locally rescaled to a parallel pure spinor with normal form of the metric given in (3.36).*

2. *There is a scale such that $||\varphi||^2 = const$. In this case, φ is a parallel $Spin^c$–CCKS on a pseudo-Kähler manifold.*

In particular, CCKS half spinors of nonzero length equivalently characterize the existence of pseudo-Kähler metrics in the conformal class.

Signature (3,2)
A real representation of $Cl_{3,2}$ on $\Delta^{\mathbb{R}}_{3,2} = \mathbb{R}^4$ is given by

$$e_1 = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}, e_2 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, e_3 = \begin{pmatrix} & -1 & & \\ -1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix},$$

$$e_4 = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, e_5 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

The complex representation on $\Delta^{\mathbb{C}}_{3,2} \cong \mathbb{C}^4$ arises by complexification and in this realisation $Spin^+(3,2) \cong Sp(2,\mathbb{R})$. The scalar product $\langle \cdot,\cdot \rangle_{\Delta^{\mathbb{C}}_{3,2}}$ is given by $\langle v,w \rangle_{\Delta^{\mathbb{C}}_{3,2}} = v^T J \overline{w}$, where $J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$. Note that $\langle v,v \rangle_{\Delta^{\mathbb{C}}_{3,2}} \in i\mathbb{R}$. Orbit representatives for the action of $Spin^c(3,2)$ on $\Delta^{\mathbb{C}}_{3,2}$ are $u := \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, u_0 := \begin{pmatrix} i & 1 & 0 & 0 \end{pmatrix}$ and $u_b := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & ib & 0 \end{pmatrix}$, where $b \in \mathbb{R}\backslash\{0\}$. One calculates that

$$\begin{aligned} \langle u,u \rangle_{\Delta^{\mathbb{C}}_{3,2}} &= 0, V_u = 0, \alpha^2_u = (e_3^\flat - e_4^\flat) \wedge (e_1^\flat - e_5^\flat), \alpha^2_u \cdot u = 0, \\ \langle u_0,u_0 \rangle_{\Delta^{\mathbb{C}}_{3,2}} &= 0, V_{u_0} = 2(e_1^\flat - e_5^\flat), \alpha^2_{u_0} = 2e_4^\flat \wedge (e_1^\flat - e_5^\flat), \alpha^2_{u_0} \cdot u_0 = 0, \\ \langle u_1,u_1 \rangle_{\Delta^{\mathbb{C}}_{3,2}} &= -i, V_{u_1} = -e_2^\flat, \alpha^2_{u_1} = (-e_1^\flat \wedge e_3^\flat + e_4^\flat \wedge e_5^\flat), \alpha^2_{u_1} \cdot u_1 = -2i \cdot u_1. \end{aligned} \tag{7.36}$$

Let $(M^{3,2},[g]=c)$ be a $Spin^c$-manifold with CCKS $\varphi \in \ker P^A$. In our local analysis, we have two cases to consider: In the first case, we find a metric $g \in c$ such that $||\varphi||^2 = \pm i$. Using (7.36) it follows exactly as in the Lorentzian $(1,4)$-case that after constantly rescaling the metric, φ is either parallel, in which case by (7.36) the metric splits into a timelike line and a pseudo-Kähler manifold, or a real or imaginary Killing spinor and V_φ, which is a timelike unit Killing vector field, defines a pseudo-Sasakian structure.
In the second case, we have that $||\varphi||^2 \equiv 0$. If φ is of orbit type $u \in \Delta^{\mathbb{R}}_{3,2}$ on an open set, it follows exactly as in the signature $(2,2)$ case that $A \equiv 0$, i.e. φ is an ordinary $Spin-$twistor spinor. The local analysis for this case has been carried out in chapter 5. Thus, we are left with the case that φ is locally of orbit type u_0. However, the analysis of this case is completely analogous to the case of Lorentzian $Spin^c$ CCKS of nonzero length and one gets a one-to-one correspondence to certain conformal Killing forms. Carrying out these steps is straightforward and we arrive at

Theorem 7.40 *Let $(M^{3,2},g)$ be a $Spin^c-$manifold of signature $(3,2)$ and let $\varphi \in \Gamma(S^g)$ be a CCKS wrt. a non-flat S^1-connection A satisfying $||\varphi||^2 \equiv 0$. Then there is a scale in which the conformal Killing form α^2_φ writes as $\alpha^2_\varphi = r^\flat \wedge V^\flat_\varphi$, where r is a spacelike vector field of constant length, V_φ is orthogonal to r and lightlike Killing and moreover*

$$d^* \alpha^2_\varphi = const. \cdot Im\langle D^A \varphi, \varphi \rangle_{S^g} \cdot V^\flat_\varphi.$$

Conversely, if $\alpha = r^\flat \wedge l^\flat$ is a conformal Killing form such that r is of constant positive length, l is lightlike and orthogonal to r and $d^*\alpha = f \cdot l^\flat$ for some function f, then there exists a nontrivial S^1-connection A and a up to S^1-action unique CCKS φ wrt. A such that $\alpha^2_\varphi = \alpha$ and $f = const. \cdot Im\langle D^A \varphi, \varphi \rangle_{S^g}$.

7.7 Remarks about the relation between conformal and normal conformal vector fields

The aim of this final subsection is to help to understand the blank relationship in the following diagram:

$$\begin{tikzcd}
\boxed{\varphi \in \ker P^{A,g} \text{ CCKS}} \arrow[r, "Spin^c \text{ to } Spin"] \arrow[d, "\text{Dirac current}"] & \boxed{\varphi \in \ker P^{g} \text{ TS}} \arrow[d, "\text{Dirac current}"] \\
\boxed{V_\varphi \text{ conformal}} \arrow[r, "???"] & \boxed{V_\varphi \text{ normal conformal}}
\end{tikzcd} \tag{7.37}$$

$Spin^c$-twistor spinors produce conformal vector fields which in general are *not* normal conformal, see (7.19). If one specializes the situation from $Spin^c$-geometry to $Spin$-geometry, we end up with twistor spinors which give rise to normal conformal vector fields. But what is the more general relation between geometries admitting conformal vector fields and geometries admitting normal conformal vector fields -not necessarily induced by spinors ? We cannot answer this in full generality but consider some illuminating examples. First, note that on a conformally flat manifold every conformal vector field is normal conformal.

Example 7.41 Consider the simply-connected, solvable Lorentzian symmetric space $M^n_{\underline{\lambda}} = (\mathbb{R}^n, g_{\underline{\lambda}})$, where

$$(g_{\underline{\lambda}})_{(s,t,x)} = 2dsdt + \sum_{j=1}^{n-2} \lambda_j x_j^2 ds^2 + \sum_{j=1}^{n-2} dx_j^2$$

and $\underline{\lambda} = (\lambda_1, ..., \lambda_{n-2})$, $\lambda_i \in \mathbb{R}\backslash\{0\}, n \geq 3$. We fix the following global orthonormal basis on $M^n_{\underline{\lambda}}$ (cf. [Bau98]):

$$\begin{aligned}
\mathfrak{a}_{\overline{0}}(y) &:= \frac{\partial}{\partial s}(y) - \frac{1}{2}\left(\sum_{j=1}^{n-2} \lambda_j x_j^2 + 1\right)\frac{\partial}{\partial t}(y), \\
\mathfrak{a}_0(y) &:= \frac{\partial}{\partial s}(y) - \frac{1}{2}\left(\sum_{j=1}^{n-2} \lambda_j x_j^2 - 1\right)\frac{\partial}{\partial t}(y), \\
\mathfrak{a}_j(y) &:= \frac{\partial}{\partial x_j}(y),
\end{aligned}$$

where $y = (s, t, x_1, ..., x_{n-2}) \in M^n_{\underline{\lambda}}$. Finally, we set $\Lambda_0 := -\sum_{j=1}^{n-2} \lambda_j$. $V := \frac{\partial}{\partial t}$ is an isotropic, parallel vector field. The Ricci tensor of $M^n_{\underline{\lambda}}$ is given by

$$Ric(X) = \Lambda_0 \cdot g(X, V)V$$

and the scalar curvature as well as the Cotton-York tensor vanish. The Weyl tensor $W := W^{g_{\underline{\lambda}}}$, considered as endomorphism of $\Lambda^2 M^n_{\underline{\lambda}}$ is given by

$$\begin{aligned}
W(\mathfrak{a}^\flat_{\overline{0}} \wedge \mathfrak{a}^\flat_j) = W(\mathfrak{a}^\flat_0 \wedge \mathfrak{a}^\flat_j) &= (\lambda_j + \frac{1}{n-2}\Lambda_0)V^\flat \wedge \mathfrak{a}^\flat_j, && j = 1, ..., n-2 \\
W(\mathfrak{a}^\flat_\alpha \wedge \mathfrak{a}^\flat_\beta) &= 0, && \text{for all other indices } \alpha, \beta.
\end{aligned}$$

Consider the Killing vector field $T(y) := \frac{\partial}{\partial s}(y)$. The causal type of T depends on the choice of the λ_i. With the above formulas, we have that the only nonvanishing contributions for $T \lrcorner W$ are given by

$$W(\mathfrak{a}_{\overline{0}}, \mathfrak{a}_j, T, \mathfrak{a}_j) = \lambda_j + \frac{1}{n-2}\Lambda_0$$

This shows that $T \lrcorner W = 0$ iff $\underline{\lambda} = (\lambda, ..., \lambda)$, i.e. T is normal conformal if and only if $M^n_{\underline{\lambda}}$ is conformally flat. In general, T does not arise as the Dirac current of a $Spin^c$ twistor spinor. On the other hand, the parallel isotropic vector field V is always normal conformal and it is the Dirac current of a parallel spinor on $M^n_{\underline{\lambda}}$.

Remark 7.42 It is also easy to think of examples of Lorentzian manifolds admitting a non-normal conformal vector field V which arises as the Dirac current of a $Spin^c$ twistor spinor φ, i.e. $V = V_\varphi$: Consider the 4-dimensional Lorentzian manifold $(\mathbb{R}^4, g)$, where we demand

$$g = \sum_{i=2}^{4} g_{1i}dx_1dx_i + \sum_{2 \leq k \leq l \leq 4} g_{kl}dx_kdx_l$$

to be a Lorentzian metric with $\frac{\partial}{\partial x_1}g_{ij} = 0$. Under this assumption $V := \frac{\partial}{\partial x_1}$ is a null Killing vector field, in particular it is conformal. [CKM+14] shows that there exists -at least locally- a connection A and a CCKS φ wrt. A such that $V_\varphi = V$. However, for generic choice of the g_{ij}, V is not normal conformal as one checks by directly computing $V \lrcorner W^g$.

As another example let us study normal conformal Killing forms on Riemannian non-Ricci-flat Einstein spaces (M, g). We have already learned in section 6.7 that they are given as sums of special Killing forms and closed conformal Killing forms. Moreover, the conformal holonomy of (M, g) coincides with that of the metric cone. Using holonomy-theory for Riemannian cones, in particular that reducible cones over complete Riemannian manifolds are automatically flat, one then proves the following classification result:

Theorem 7.43 ([Lei05]) *Let (M^n, g) be a simply-connected and complete Riemannian Einstein space of positive scalar curvature admitting a normal conformal Killing p–form. Then M^n is either*

1. *The round (conformally flat) sphere S^n.*
2. *an Einstein-Sasaki manifold of odd dimension $n \geq 5$ with a special Killing 1-form.*
3. *an Einstein-3-Sasaki space of dimension $n = 4m+3 \geq 7$ with three independent special Killing 1-forms.*
4. *a nearly Kähler manifold of dimension 6, where the Kähler form is a special Killing 2-form.*
5. *a nearly parallel G_2-manifold in dimension 7 with its fundamental form a special Killing 3-form.*

If the space is not complete and the cone reducible then the metric g has up to a constant scaling factor locally the form

$$dt^2 + sin^2(t) \cdot k + cos^2(t) \cdot h, \tag{7.38}$$

where k^p and h^q are arbitrary Riemannian Einstein metrics of positive scalar curvature on spaces with dimension p resp. q. The scaled volume forms $sin^{-p}\cdot dvol_k$ and $cos^{-q}\cdot dvol_h$ are special Killing of degree p resp. q for (M,g).

Consequently, we see that among the non-conformally flat, complete Riemannian Einstein space of positive scalar curvature, the difference between conformal vector fields and normal conformal vector fields corresponds by the cases 2. and 3. of the preceding Theorem precisely to the difference between conformal vector fields and Killing vector fields inducing Sasakian structures which is an answer for the ???-direction in diagram (7.37) for these geometries. However, we do not know how to complete this blank direction in diagram (7.37) in general.

Remark 7.44 Given a generic conformal vector field on an Riemannian Einstein space, one can produce a normal conformal vector field out of it. More precisely, [KR94] shows that given a conformal vector field V on a Riemannian Einstein space which is neither isometric nor homothetic, then the vector field $grad(divV)$ is (locally) a conformal gradient field on a warped product of type (7.38), where $p=0$ or $q=0$. In this description, $grad(divV)$ is the Hodge dual of the normal conformal Killing form $sin^{-p}\cdot dvol_k$ resp. $cos^{-q}\cdot dvol_h$, and therefore itself a normal conformal Killing field , cf. [Lei05].

Remark 7.45 It is interesting to observe that the difference between conformal and normal conformal tensors vanishes in the spinorial $Spin$–setting. More precisely, if $\varphi\in\Gamma(M,S^g)$ is a $Spin$–twistor spinor, then it *always* satisfies the *normalization condition* $\nabla_X D^g\varphi=\frac{n}{2}K^g(X)\cdot\varphi$, ensuring that $\left(\widetilde{\Phi}^g\right)^{-1}\begin{pmatrix}\varphi\\ -\frac{1}{n}D^g\varphi\end{pmatrix}\in\Gamma(M,\mathcal{S})$ is a parallel spin tractor. There is no notion of normal conformal Killing spinors. On the other hand, a conformal Killing form does in general not satisfy the additional normalisation conditions needed to give a parallel tractor form. However, conformal, not necessarily normal conformal, Killing forms can be described in terms of the tractor machinery as well, using the theory of first BGG-operators, see [Ham08]. One shows that there is a unique modified connection

$$\widetilde{\nabla}^{nc}:\Gamma(M,\Lambda^*_{\mathcal{T}}(M))\to\Gamma(T^*M\otimes\Lambda^*_{\mathcal{T}}(M)),$$
$$\widetilde{\nabla}^{nc}=\nabla^{nc}+\Theta,$$

where $\Theta\in\Omega^1\left(M,End\left(\Lambda^*_{\mathcal{T}}(M)\right)\right)$ such that conformal Killing forms are equivalently characterized as tractors $\alpha\in\Omega^*_{\mathcal{T}}(M)$ satisfying

$$\widetilde{\nabla}^{nc}\alpha=0. \tag{7.39}$$

Note that we have derived an analogue of (7.39) for $Spin^c$-twistor spinors in Theorem 7.11. They are also characterized as parallel spin tractors wrt. a modified Cartan connection, given by $\nabla^{nc}-F_{dA}$. In this sense, passing from normal conformal Killing forms to conformal Killing forms corresponds to passing from $Spin$–twistor spinors to $Spin^c$–twistor spinors on the spinorial level.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig und nur unter Verwendung der angegebenen Hilfsmittel angefertigt habe.

Ich erkläre, dass ich die Arbeit erstmalig und nur an der Humboldt-Universität zu Berlin eingereicht habe und mich nicht andernorts um einen Doktorgrad beworben habe. In dem angestrebten Promotionsfach besitze ich keinen Doktorgrad.

Der Inhalt der dem Verfahren zugrunde liegenden Promotionsordnung ist mir bekannt.

Berlin, den 14. Juli 2014 Andree Lischewski